February 9, 2008

# 8-VERTEX CORRELATION FUNCTIONS
# AND
# TWIST COVARIANCE OF q-KZ EQUATION

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*ABSTRACT.* We study the vertex operators $\Phi(z)$ associated with standard quantum groups. The element $Z = RR^{\rm t}$ is a "Casimir operator" for quantized Kac-Moody algebras and the quantum Knizhnik-Zamolodchikov (q-KZ) equation is interpreted as the statement $:Z\Phi(z): = \Phi(z)$. We study the covariance of the q-KZ equation under twisting, first within the category of Hopf algebras, and then in the wider context of quasi Hopf algebras. We obtain the intertwining operators associated with the elliptic R-matrix and calculate the two-point correlation function for the eight-vertex model.

Mathematics Subject Classification (1991) 81R50, 70G50.

# 1. Introduction.

In this paper we study the quantum Knizhnik-Zamolodchikov equation [FR] for quasi Hopf algebras, with its covariance properties with respect to twisting, and its relation to matrix elements of intertwining operators. The conclusions bear on the interpretation of the solutions of similar equations with exotic R-matrices. We calculate the correlation functions for the 8-vertex model.

*Correlation Functions for the Eight-Vertex Model.*

Baxter [Ba] introduced the trigonometric and elliptic quantum R-Matrix for $\widehat{\mathfrak{sl}(2)}$; this paper is mostly about the elliptic case, and about the generalization [Be] to elliptic quantum $\widehat{\mathfrak{sl}(N)}$. The trigonometric R-matrices found their interpretation in terms of quantized Kac-Moody algebras, viewed as Hopf algebras; that is, quantum groups [D1]. The elliptic R-matrices had, until recently, not found their place in an algebraic framework. Surprisingly the elliptic R-matrices also turned out to be related to quantized Kac-Moody algebras, but with a quasi Hopf structure [Fr1,2]. More precisely, the algebraic structure is the same as in the trigonometric case, while the coproduct $\Delta$ of the trigonometric quantum group is replaced by a new, deformed coproduct $\Delta_\epsilon$ ("elliptic coproduct") that depends on a deformation parameter $\epsilon$. It can be expressed as $\Delta_\epsilon = (F_\epsilon^{\rm t})^{-1}\Delta F_\epsilon^{\rm t}$; the twistor $F_\epsilon$ must satisfy a cocycle condition that has been solved to give an explicit expression for $F_\epsilon$ as a power series in $\epsilon$. The quotient of the elliptic quantum group, by the ideal generated by the center, is a Hopf algebra; it is the quantization, in the sense of Drinfel'd, of the classical, affine Lie bialgebra with elliptic r-matrix in the classification of Belavin and Drinfeld [BD].

To understand the role of these elliptic quantum groups in the context of integrable models and conformal field theory, we calculate the correlation functions of the eight-vertex model. The premise is that Baxter's vertex operators can be interpreted mathematically as intertwining operators for representations of quantized Kac-Moody algebras [JM]; this is the interpretation that affords the most direct link between statistical models and conformal field theory. Here we define new intertwining operators in terms of the elliptic coproduct and calculate the correlation functions that are associated with them; that is, matrix elements of products of intertwining operators. We find that these functions satisfy equations similar to the quantum Knizhnik-Zamolodchikov equations of Frenkel and Reshetikhin [FR], but that they can be described much more easily in terms of the familiar correlation functions that govern the six-vertex model.

*Twist Covariance.*

The larger issue is the question of the covariance of the q-KZ equation under twisting in the category of quasi Hopf algebras. To begin with, we point out that the q-KZ of Frenkel and Reshetikhin [FR] can be easily generalized to all simple, affine quantum groups endowed with what we call a "standard" R-matrix: a universal R-matrix (expressed as a series in Chevalley-Drinfeld generators, see Definition 2.1.) that commutes with the Cartan subalgebra. Reshetikhin [R] has described a highly specialized form of twisting under which a standard R-matrix remains of standard type. From now on, by the term "twisting" we always have in mind a more radical twist that transforms a standard R-matrix to a nonstandard or esoteric R-matrix.

A quantum group in the sense of this paper is a quantized, affine Kac-Moody algebra $\hat{\mathfrak{g}}$ based on a simple Lie algebra $\mathfrak{g}$. The structure of coboundary Hopf algebra is given by a coproduct, an antipode and a counit, but only the coproduct plays a direct role in this paper. A coboundary Hopf algebra is a Hopf algebra $\hat{\mathfrak{g}}$ with an invertible element $R \in \hat{\mathfrak{g}} \otimes \hat{\mathfrak{g}}$ that satisfies the Yang-Baxter relation and that intertwines the coproduct $\Delta$ with its opposite $\Delta'$:

$$R\Delta' = \Delta R. \tag{1.1}$$

The q-KZ equation is a holonomic system of difference equations that are satisfied by certain intertwining operators,

$$\Phi, \Psi : V_{\mu,k} \to V(z) \otimes V_{\nu,k}, \tag{1.2}$$

where $V_{\mu,k}$ and $V_{\nu,k}$ are irreducible, highest weight $\hat{\mathfrak{g}}$-modules of level $k$ and $V(z)$ is an evaluation module. The intertwining property of $\Phi$ and of $\Psi$ is expressed as

$$\Phi x = \Delta(x)\Phi, \quad \Psi x = \Delta'(x)\Psi, \tag{1.3}$$

for $x \in \hat{\mathfrak{g}}$. When $R$ is of standard type (Definition 2.1), then the q-KZ equation for $\Psi$ takes the form

$$(Z' - 1)\Psi = 1, \tag{1.4}$$

where $Z'$ is a Casimir operator (acting in $V(z) \otimes V_{\nu,k}$) for $\hat{\mathfrak{g}}$. To define this operator let us express $R$ as

$$R = R^i \otimes R_i,$$

where we use the summation convention for the index $i$; then formally,

$$Z' = R^{\mathrm{t}} R, \quad R^{\mathrm{t}} := R_i \otimes R^i. \tag{1.5}$$

However, to make sense of an operator product such as $Z'\Psi$ it is necessary to renormalize it. The correct form of the q-KZ equation is indeed (1.4), but with $Z'\Psi$ replaced by the normal-ordered product

$$:Z'\Psi: = R^{\mathrm{t}}(R^i q^{\hat{H}} \otimes 1)\Psi R_i, \tag{1.6}$$

where the factor $q^{\hat{H}}$ belongs to the Cartan subalgebra of $\hat{\mathfrak{g}}$.

We study a deformation of the initial, standard quantum group, implemented by twisting with an invertible element $F_\epsilon \in \hat{\mathfrak{g}} \otimes \hat{\mathfrak{g}}$ that is a formal power series in a deformation parameter $\epsilon$. The twisted quantities are

$$\begin{aligned} R_\epsilon &= (F_\epsilon^{\mathrm{t}})^{-1} R F_\epsilon, \quad \Delta'_\epsilon = F_\epsilon \Delta' F_\epsilon^{-1}, \\ \Psi_\epsilon &= F_\epsilon^{-1}\Psi, \quad Z'_\epsilon = F_\epsilon^{-1} Z' F_\epsilon, \end{aligned}$$

and the twisted KZ equation is

$$:Z'_\epsilon \Psi_\epsilon: = \Psi_\epsilon;$$

it has the same form as in the standard case. However, Eq.(1.6) is not covariant; we mean by that it cannot be generalized by simply replacing $R$ by $R_\epsilon$, since the expression

$$R_\epsilon^{\mathrm{t}}(R_\epsilon^i \otimes 1)\Psi_\epsilon R_{\epsilon i}$$

is not well defined. Instead, the correct expression for the normal-ordered product is

$$:Z'_\epsilon \Psi_\epsilon: = F_\epsilon^{-1} :Z'\Psi: = F_\epsilon^{-1} R^{\mathrm{t}}(R^i q^{\hat{H}} \otimes 1)\Psi R_i.$$

Therefore, though there is a clear sense in which "the q-KZ equation" is covariant, the normal-ordered product (1.6) is not.

This observation has analogous implications for correlation function. To illustrate this, consider the two-point correlation function $g(z_1, z_2) = \langle \Psi(z_1)\Psi(z_2)\rangle$. In the standard case the q-KZ equation reduces to

$$g(q^{-k-g} z_1, z_2) = q^{A_1} R^{-1}(z_1, z_2) g(z_1, z_2). \tag{1.7}$$

The twisted correlation function obeys

$$g_\epsilon(q^{-k-g}z_1, z_2) = \big(F_\epsilon^{-1}(z_2, q^{-k-g}z_1)q^{A_1}R^{-1}(z_1,z_2)F_\epsilon(z_2,z_1)\big)g_\epsilon(z_1,z_2),$$

and this is *not* the same as Eq.(1.7) with $R$ replaced by $R_\epsilon$.

This conclusion casts some light on the proposed generalization of of the q-KZ equations for correlation functions. Integrability is assured by the Yang-Baxter relation for the R-matrix. It is natural to study the equations that result from replacing the trigonometric R-matrix in (1.7) and the rest, by more exotic R-matrices. Since this requires a knowledge of such R-matrices in finite dimensional representations only, it is possible, in particular, to use the elliptic R-matrix of Baxter in this connection. As long as the elliptic quasi Hopf algebra was not known, it was possible to speculate that the solutions of such "elliptic q-KZ equations" relate in some way to (unknown) elliptic intertwiners. Our conclusion is that this interpretation is not the correct one.

*Outline of the paper.*

Section 2 summarizes some facts about standard, universal R-matrices and sets our notation. Section 3 examines certain intertwining operators and draws some conclusions (Proposition 3.1) that are used later to determine the correct approach to regularizing operator products.

Sections 4 and 5 present a view of the KZ and q-KZ equations. Both can be interpreted very simply as eigenvalue equations, $\zeta\Phi = 0$ or $(Z-1)\Phi = 0$, for the Casimir operators $\zeta$ or $Z$ of affine Kac-Moody or quantized, affine Kac-Moody algebras. Section 4 deals with the classical KZ equation $\zeta\Phi = 0$; the effect of different polarizations is discussed, as well as the invariance of the operator $\zeta$ (Propositions 4.1 and 4.2). The quantum case is taken up in Section 5; the correct normal-ordered action of the Casimir elements $Z$ and $Z'$ on the intertwiners $\Phi$ and $\Psi$ is established (Proposition 5.1), and the q-KZ equations are presented in Eq.s (5.6) and (5.8).

Sections 6 and 7 explore the effect on intertwiners of twisting in the categories of Hopf and quasi Hopf algebras. In Section 6 we stress the distinction between "finite" and "elliptic" twisting. The twisted q-KZ equation is presented (Definition 6.3). In Section 7 quasi Hopf twisting is discussed and a recursion relation to actually calculate the elliptic twistor is given.

Sections 8 and 9 apply the results to correlation functions. In Section 8 the classical and quantum q-KZ equations for correlation functions are given; the effect of twisting is exhibited and a certain lack of covariance is emphasized. In Section 9 the two-point correlation function for the eight-vertex model is calculated, as well as explicit expressions for the twisting matrix in the fundamental representation of $\widehat{\mathfrak{sl}(2)}$.

Finally, some auxiliary material is relegated to an Appendix.

*Relation to other work.*

(1) Our original goal was to discover the enigmatic "elliptic quantum groups" and to use it to define and to calculate the correlation functions for the eight-vertex model. This is precisely the problematics of a series of paper by Jimbo, Miwa and others; see especially the review [JM] and the paper [JMN]. These authors did not have available the universal, elliptic R-matrix and did not anticipate the fact that the algebraic structure of the elliptic quantum group would turn out to be the same as in the trigonometric case. (Only the coproduct is changed.) They postulated a new algebraic structure, but in the absence of a coproduct they could not define intertwiners. In spite of this they did succeed in calculating correlation functions that stand up to analysis and that reproduce some of Baxter's results on the 8-vertex model. Nevertheless, the correlation functions that we here propose for the eight-vertex model are quite different.

(2) One of the most interesting aspects of the elliptic quantum group is its quasi Hopf nature. Quasi Hopf algebras, characterized by a modified quantum Yang-Baxter relation, are basic to

the Knizhnik-Zamolodchikov-Bernard generalization of the KZ equation that was discovered by Bernard [Ber]. This equation also arises in connection with Felder's elliptic quantum groups [Fe]. However, these developments are not concerned with highest weight matrix elements of intertwiner operators, and the quasi Hopf algebras of Felder *et al.* are not related to the elliptic R-matrices of Baxter and Belavin. The new r-matrices discovered by Enriquez and Rubtsov [ER] and by Frenkel, Reshetikhin and Semenov-Tian-Shansky [FRS] are of a different sort. These interesting developments go beyond the classification of classical r-matrices by Belavin and Drinfel'd [BD] and are outside the scope of this paper.

## 2. Standard, affine, universal, quantum R-matrices.

This section contains basic definitions and notation.

Let $M, N$ be two finite sets, $\varphi, \psi$ two maps,

$$\begin{aligned}\varphi : M \times M \to \mathbb{C}, \quad & a, b \mapsto \varphi^{ab},\\ \psi : M \times N \to \mathbb{C}, \quad & a, \beta \mapsto H_a(\beta),\end{aligned}$$

and $q$ a complex parameter. Let $\mathcal{A}$ or $\mathcal{A}(\varphi, \psi)$ be the universal, associative, unital algebra over $\mathbb{C}$ with generators $\{H_a\}_{a\in M}, \{e_{\pm\alpha}\}_{\alpha\in N}$, and relations

$$\begin{aligned}&[H_a, H_b] = 0\ , \quad [H_a, e_{\pm\beta}] = \pm H_a(\beta) e_{\pm\beta},\\ &[e_\alpha, e_{-\beta}] = \delta_\alpha^\beta\big(q^{\varphi(\alpha,\cdot)} - q^{-\varphi(\cdot,\alpha)}\big),\end{aligned}$$

with $\varphi(\alpha,\cdot) = \varphi^{ab}H_a(\alpha)H_b, \varphi(\cdot,\alpha) = \varphi^{ab}H_aH_b(\alpha)$ and $q^{\varphi(\alpha,\cdot)+\varphi(\cdot,\alpha)} \neq 1, \alpha \in N$. The algebra of actual interest is a quotient $\mathcal{A}' = \mathcal{A}/\mathcal{I}$ where $\mathcal{I}$ is a certain ideal; in this paper we suppose that $\mathcal{I}$ is generated by a complete set of (quantized) Serre relations among the $e_\alpha$'s and among the $e_{-\alpha}$'s; then $\mathcal{A}'$ is a quantized (generalized) Kac-Moody algebra. In the case when $\mathcal{A}'$ is a quantized Kac-Moody algebra of affine type, based on a simple Lie algebra $\mathfrak{g}$, we sometimes write $\hat{\mathfrak{g}}$ for $\mathcal{A}'$. The "Cartan subalgebra" $\mathcal{A}'_0$ is generated by $\{H_a\}_{a\in M}$, extended by the inclusion of exponentials.

**Definition 2.1.** The standard, universal R-matrix has the form

$$R = q^\varphi T = q^\varphi \sum_{n=0}^\infty t_n, \quad \varphi = \sum \varphi^{ab} H_a \otimes H_b, \tag{2.1}$$

where $t_0 = 1 \otimes 1, t_1 = \sum e_{-\alpha} \otimes e_\alpha$ (the sum is over the Serre generators, $\alpha \in N$) and $t_n$ has the form

$$t_n = t_{(\alpha)}^{(\alpha')} e_{-\alpha_1} ... e_{-\alpha_n} \otimes e_{\alpha'_1} ... e_{\alpha'_n}. \tag{2.2}$$

Sums over repeated indices are implied; the multi-index $(\alpha')$ runs over the permutations of $(\alpha)$.

The coefficients $t_{(\alpha)}^{(\alpha')} \in \mathbb{C}$ are essentially determined (the elements $t_n$ are determined uniquely) by the imposition of the Yang-Baxter relation,

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{2.3}$$

It has been shown that, for a universal R-matrix of the type (2.1), this relation is equivalent to the recursion relation [Fr1]

$$[e_\gamma \otimes 1, t_n] = t_{n-1}(q^{\varphi(\gamma,.)} \otimes e_\gamma) - (q^{-\varphi(.,\gamma)} \otimes e_\gamma)t_{n-1}, \tag{2.4}$$

with the initial condition $t_0 = 1$. There is exactly one solution in $\mathcal{A}' \otimes \mathcal{A}'$.

We suppose now that $\mathcal{A}' = \hat{\mathfrak{g}}$ is a quantized, affine Kac-Moody algebra based on a simple Lie algebra $\mathfrak{g}$. The coproduct is then generated by the following formulas,

$$\Delta(e_\alpha) = 1 \otimes e_\alpha + e_\alpha \otimes q^{\varphi(\alpha,.)}, \quad \Delta(e_{-\alpha}) = q^{-\varphi(.,\alpha)} \otimes e_{-\alpha} + e_{-\alpha} \otimes 1, \tag{2.5}$$

and $\Delta H_a = H_a \otimes 1 + 1 \otimes H_a$. Let $\pi_1, \pi_2$ be finite dimensional representations of $\mathfrak{g}$, and $\pi_i(z_i)$ the associated evaluation representations of $\hat{\mathfrak{g}}$ with spectral parameters $z_i$. Let

$$R(z_1, z_2) := \pi_1(z_1) \otimes \pi_2(z_2)\ R. \tag{2.6}$$

The spectral parameters are regarded as formal variables; $R(z_1, z_2)$ is a formal power series $R_{12}(z_2/z_1)$ in $z_2/z_1$. The effectiveness of the recursion relation (2.4) is illustrated in the Appendix.

Finally, given $A = a^i \otimes a_i \in \mathcal{A}' \otimes \mathcal{A}'$, we shall write $A^{\mathrm{t}} := a_i \otimes a^i$ and $\mathrm{m}A := a^i a_i$.

# 3. Highest weight modules and intertwining operators.

Let $V_\mu$ be an irreducible, finite dimensional, highest weight $\hat{\mathfrak{g}}$-module, and $V_{\mu,k} = \bigoplus_{n\geq 0} V_{\mu,k}[-n]$ the associated level $k$, highest weight, irreducible, graded $\hat{\mathfrak{g}}$-module. The intertwining operators of greatest interest are imbeddings

$$\Phi = \Phi(z) : V_{\mu,k} \to V(z) \otimes V_{\nu,k}, \tag{3.1}$$

where $V(z)$ is an evaluation module over $\hat{\mathfrak{g}}$. The defining property of $\Phi$ is

$$\Phi x = \Delta(x)\Phi,$$

for all $x$ in $\hat{\mathfrak{g}}$.

We shall obtain some very essential information about the structure of the intertwining operators.

**Proposition 3.1.** *Let $v$ be a homogeneous element of $V_{\mu,k}$. Then $\Phi v = \sum_n a_n \otimes b_n$, with $b_n \in V_{\nu,k}[-n]$, where the sum is not, in general, finite.*

*Proof.* It will be enough to verify that the sum is effectively infinite in one typical case. Thus consider the quantized Kac-Moody algebra $\widehat{\mathfrak{sl}(2)}$ with $V$ the fundamental representation. In this case, for any $v \in V_{\mu,k}$, $\Phi v$ takes the form

$$\Phi v = \begin{pmatrix} A(z)v \\ B(z)v \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} v. \tag{3.2}$$

Necessary conditions to be satisfied by the operators $A, B : V_{\mu,k} \to V_{\nu,k}$ are

$$\Delta(e_\alpha) \begin{pmatrix} A \\ B \end{pmatrix} v = \begin{pmatrix} A \\ B \end{pmatrix} e_\alpha v. \tag{3.3}$$

The first space is two-dimensional, with

$$e_1 \otimes 1 = \kappa \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_0 \otimes 1 = \kappa \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix}. \tag{3.4}$$

The parameter $\kappa$ is related to $q$, $\kappa^2 = q - q^{-1}$. In full detail,

$$\begin{aligned} &[e_0, A] = 0, \quad [e_1, A] = -\kappa q^{\varphi(1,.)} B, \\ &[e_1, B] = 0, \quad [e_0, B] = -\kappa z q^{\varphi(0,.)} A. \end{aligned} \tag{3.5}$$

On the highest weight vector $v_0$ in $V_{\mu,k}$ we have

$$e_0 A v_0 = 0, \quad e_1 B v_0 = 0, \quad e_0 B v_0 = -\kappa z q^{\varphi(0,.)} A v_0, \quad e_1 A v_0 = -\kappa q^{\varphi(1,.)} B v_0, \tag{3.6}$$

with a unique solution of the form

$$A v_0 = \sum_{n=0}^{\infty} z^n v'_{2n}, \quad B v_0 = \sum_{n=0}^{\infty} z^{n+1} v'_{2n+1}, \tag{3.7}$$

with $v'_0$ a highest weight vector in $V_{\nu,k}$ and vectors $v'_n \in V_{\nu,k}$ determined recursively by

$$e_0 v'_{2n} = e_1 v'_{2n+1} = 0, \quad e_1 v'_{2n} = -\kappa q^{\varphi(1,.)} v'_{2n-1}, \quad e_0 v'_{2n+1} = -\kappa q^{\varphi(0,.)} v'_{2n}. \tag{3.8}$$

The solutions have the form

$$v'_{2n} = \sum_{\sigma \in S_{2n}} A^{2n}_\sigma \, \sigma(e_{-1} e_{-0})^n v'_0, \quad v'_{2n+1} = \sum_{\sigma \in S_{2n+1}} B^{2n+1}_\sigma \sigma(e_{-1} e_{-0})^n e_{-0} v'_0,$$

where the sum is over all permutations of the generators. It is clear that $v'_n \neq 0$ for all $n$ and the proposition is proved.

We return to the general case, $\mathcal{A}' = \hat{\mathfrak{g}}$ is a quantized Kac-Moody algebra of affine type, based on a simple Lie algebra $\mathfrak{g}$, $V_{\mu,k}$ is a highest weight module over $\hat{\mathfrak{g}}$, $V_i(z_i)$ finite dimensional evaluation modules.

*Remark 3.2.* The product $\Phi_2 \Phi_1$ is a compound map

$$\Phi_2 \Phi_1 : \; V_{\mu,k} \xrightarrow{\Phi_1} V_1 \otimes V_{\nu,k} \xrightarrow{\Phi_2} V_1 \otimes V_2 \otimes V_{\lambda,k}. \tag{3.9}$$

It has the property

$$\Phi_2 \Phi_1 x = \Phi_2 \Delta(x) \Phi_1 = (\mathrm{id} \otimes \Delta) \Delta(x) \Phi_2 \Phi_1. \tag{3.10}$$

By coassociativity of $\Delta$, $\Phi_2 \Phi_1$ is an intertwiner of the same type as $\Phi_1$ and $\Phi_2$:

$$\Phi_2 \Phi_1 : V_{\mu,k} \to \big(V_1(z_1) \otimes V_2(z_2)\big) \otimes V_{\lambda,k}. \tag{3.11}$$

Consequently, universal statements about intertwiners apply to products of intertwiners as well. This observation will be of use in Section 8. Of course, it does not apply in the quasi Hopf case (Section 9.)

# 4. The classical KZ equation.

The object

$$Z = RR^{\rm t} \in \mathcal{A}' \otimes \mathcal{A}', \tag{4.1}$$

if it exists, is invariant in the sense that it commutes with $\Delta(x), \forall x \in \mathcal{A}'$. It plays the role of a Casimir element for the quantized Kac-Moody algebra. Since the intertwiner $\Phi$ projects on an irreducible representation, one expects that there is $\langle Z \rangle \in \mathbb{C}$ such that

$$(Z - \langle Z \rangle)\Phi = 0. \tag{4.2}$$

We shall begin our study of this equation by considering its classical limit. The result is Propositions (4.2) and (4.3). The important concepts are normal ordering and "polarization". Then we shall return to the quantum case to show that (4.2) is the q-KZ equation of Frenkel and Reshetikhin [FR]. (Section 5.)

The classical limit is defined by setting $q = {\rm e}^\eta$, expanding in powers of $\eta$, and retaining the first nonvanishing term. When $\mathcal{A}' = \hat{\mathfrak{g}}$ is a quantized Kac-Moody algebra of finite type, one finds that

$$R = 1 + \eta r + O(\eta^2), \quad r = \varphi + \sum_{\alpha \in \Delta^+} E_{-\alpha} \otimes E_\alpha, \tag{4.3}$$

where the sum runs over the positive roots of $\mathfrak{g}$. For simple roots one has $e_\alpha = \sqrt{\eta}(E_\alpha + O(\eta))$; the others are normalized so that the Casimir element in $\mathfrak{g} \otimes \mathfrak{g}$ takes the form

$$\mathcal{C} = r + r^{\rm t}.$$

In the case of an untwisted affine loop algebra one gets

$$R = 1 + \eta r + O(\eta^2), \quad r = \varphi + \sum_{\alpha \in \Delta^+} E_{-\alpha} \otimes E_\alpha \ + \sum_{n \geq 1} (z_2/z_1)^n \mathcal{C}, \tag{4.4}$$

where $\Delta^+$ is the set of positive roots of the underlying Lie algebra and where $z_1, z_2$ are the spectral parameters in the first, resp. second space. It is important to keep in mind that this expression is, until further development, nothing more than a formal power series in $z_2/z_1$. In terms of the basis

$$E^n_{\pm\alpha} = z^n E_{\pm\alpha}, \quad H^n_a = z^n H_a, \tag{4.5}$$

the expression for $r$ becomes

$$r = \varphi + E_{-\alpha} \otimes E_\alpha + \sum_{n \geq 1} \mathcal{C}^n, \tag{4.6}$$

with

$$\mathcal{C}^n = K^{ab} H^{-n}_a \otimes H^n_b + E^{-n}_{-\alpha} \otimes E^n_\alpha + E^{-n}_\alpha \otimes E^n_{-\alpha}, \quad K^{ab} = (\varphi + \varphi^{\rm t})^{ab}. \tag{4.7}$$

Summation over $a, b$ and $\alpha \in \Delta^+$ will henceforth be taken for granted. Note that Eq.s (4.6) and (4.7) are valid in the case of twisted loop algebras as well.

Returning to affine Kac-Moody algebras, it will be convenient to change our conventions just a little. Retain the above notation for the loop algebra, so that, in particular,

$$\varphi = \varphi^{ab} H_a \otimes H_b, \tag{4.8}$$

where the sum runs over the basis of the Cartan subalgebra of a simple Lie algebra $\mathfrak{g}$. The form that characterizes the full, quantized affine Kac-Moody algebra $\hat{\mathfrak{g}}$ is

$$\hat{\varphi} = \varphi + uc \otimes d + (1-u)d \otimes c, \tag{4.9}$$

where $d$ is the degree operator, $c$ is a basis for the central extension and $u$ is a parameter. For the full quantized Kac-Moody algebra the limit is

$$R = 1 + \eta\hat{r} + O(\eta^2), \quad \hat{r} = r + uc \otimes d + (1-u)d \otimes c. \tag{4.10}$$

The classical limit of $Z$ is

$$Z = 1 + \eta\zeta + O(\eta^2), \quad \zeta = \hat{r} + \hat{r}^{\mathrm{t}}. \tag{4.11}$$

Formally,

$$\zeta = \hat{r} + \hat{r}^{\mathrm{t}} = \sum_{-\infty}^{\infty} \mathcal{C}^n + c \otimes d + d \otimes c.$$

When both spaces are evaluation modules, where $c \mapsto 0$,

$$\zeta = \sum_{n=-\infty}^{+\infty} (z_2/z_1)^n \mathcal{C}. \tag{4.12}$$

This sum becomes zero when projected on a quotient algebra of meromorphic functions.

We try to make sense out of the classical limit of (4.2), namely

$$(\zeta - \langle\zeta\rangle)\Phi = 0.$$

By abuse of notation we retain the notation $\Phi$ for the classical limit of the intertwiner. Now the first space is an evaluation mode, where $c$ vanishes, and if $c \mapsto k$ ($k$ is the level) on the second space, then formally

$$\zeta\Phi(z) = kz\frac{\mathrm{d}}{\mathrm{d}z}\Phi(z) + \sum_{n\geq 0} \mathcal{C}^{-n}\Phi(z) + \sum_{n>0} \mathcal{C}^n\Phi(z). \tag{4.13}$$

Let us introduce a uniform basis $\{L_a\}$ for $\mathfrak{g}$, so that the Casimir element takes the form $\mathcal{C} = L_a \otimes L_a$ (summation implied). Then (in the untwisted case) (4.13) takes the form

$$\zeta\Phi(z) = \Big(kz\frac{\mathrm{d}}{\mathrm{d}z} + L_a \otimes \sum_{n\geq 0} z^n L_a^{-n} + L_a \otimes \sum_{n>0} z^{-n} L_a^n\Big)\Phi(z). \tag{4.14}$$

However, the significance of this formula is doubtful, as we shall see. This is the reason for the introduction of normal-ordered products in [FR].

*Polarization.*

It is usual, at this point of the development, to replace the operator products by normal-ordered products. It is a step that merits comment. Normal-ordered operator products are

introduced in field theory when ordinary operator products fail to make sense. The typical example is this product of destruction and creation operators:

$$\Big(\sum_n \mathrm{e}^{\mathrm{i}n\omega} a_n\Big)\Big(\sum_m \mathrm{e}^{-\mathrm{i}m\omega} a_m^*\Big).$$

When it is applied to the vacuum one gets

$$\Big(\sum_n \mathrm{e}^{\mathrm{i}n\omega} a_n\Big)\Big(\sum_m \mathrm{e}^{-\mathrm{i}m\omega} a_m^*\Big)|0\rangle = \sum_{n=-\infty}^{+\infty} |0\rangle,$$

which is without meaning. The last term in (4.14) is of this kind; the degree-decreasing operators in $\Phi$ correspond to the creation operators, and the degree-increasing operators $L_a^n$ correspond to the destruction operators. Collecting all terms of the same degree in the product one gets a divergent series. We want to avoid having to interpret such infinite series, if it is possible.

Using the (classical) intertwining property of the intertwiner,

$$\sum_{n=1}^{N} z^{-n}(L_a \otimes L_a^n)\Phi = \sum_{n=1}^{N} z^{-n}(L_a \otimes 1)\Phi L_a^n - \Big(\sum_{n=1}^{N} L_a L_a \otimes 1\Big)\Phi. \tag{4.15}$$

Passing with $N$ to infinity we encounter the meaningless expression $\sum_{n>0} L_a L_a$, an exact analogue of the divergent sum that is thrown away when a field operator product is replaced by the normal-ordered product. It is tempting to redefine the operator $\zeta$, by dropping this offensive term, thus

$$\zeta\Phi = kz\frac{\mathrm{d}}{\mathrm{d}z}\Phi + (L_a \otimes 1){:}J_a(z)\Phi{:} \quad (?), \tag{4.16}$$

with

$$J_a = J_a^+ + J_a^- = \sum_{n\geq 0} z^n L_a^{-n} + \sum_{n>0} z^{-n} L_a^n, \tag{4.17}$$

and

$${:}J_a\Phi{:} := J_a^+\Phi + \Phi J_a^-. \tag{4.18}$$

This new operator is well defined on highest weight modules and it will serve if it has the property that the formal expression (4.1) was intended to assure; that is, if it is invariant. Actually it is, almost.

**Proposition 4.1.** *The covariant definition of the operator product $\zeta\Phi(z)$ is*

$$\zeta\Phi(z) = (k+g)z\frac{\mathrm{d}}{\mathrm{d}z}\Phi(z) + (L_a \otimes 1){:}J_a(z)\Phi(z){:}, \tag{4.19}$$

*where $g$ is the dual Coxeter number of $\mathfrak{g}$.*

This result of [FR] is an analogue of Proposition 4.2 that we prove below. The replacement of the factor $k$ by $k+g$, at first sight somewhat mysterious, is thus required by covariance. For $\mathfrak{sl}(N)$, $g = N$.

We calculate the value $\langle\zeta\rangle$. The operator $J_-$ annihilates the highest weight vector $v_0$; therefore

$$\zeta\Phi(z)v_0 = \Big((k+g)z\frac{\mathrm{d}}{\mathrm{d}z} + L_a \otimes J_a^+\Big)\Phi(z)v_0.$$

In terms of the contravariant bilinear form (.,.), with $v_0'$ the highest weight vector of $V_{\nu,k}$, one gets a $V(z)$-valued function

$$(v_0',\, \Phi(z)v_0) =: \Phi_{v_0'v_0}(z) \in V(z),$$

and

$$\big(v_0',\, \zeta\Phi(z)v_0\big) = (k+g)z\frac{\mathrm{d}}{\mathrm{d}z}\Phi_{v_0'v_0}(z) + (v_0', L_a \otimes J_a^+\ \Phi(z)v_0).$$

In $J_i^+$ only the zero mode contributes, and the second term reduces to const.$\times\Phi_{v_0'v_0}(z)$. The constant has the value

$$\tfrac{1}{2}\big(C(\mu) - C(\nu) - C(\pi)\big), \tag{4.20}$$

where $C(\mu)$ is the value of the Casimir operator $C = \mathrm{m}\,\mathcal{C}$ in $V_{\mu,k}[0]$. (Recall that if $A = a \otimes b \in \mathcal{A}' \otimes \mathcal{A}'$, then $\mathrm{m}A = ab \in \mathcal{A}'$.)

We can reduce the value $\langle\zeta\rangle$ of $\zeta$ to zero by choosing the grading of $\Phi$ according to

$$\Phi(z) = \sum_{n\in\mathbf{Z}} \Phi[n] z^{-n-(\mu|v\pi)}, \quad (\mu|\nu,\pi) := \frac{1}{2(k+g)}\big(C(\mu) - C(\nu) - C(\pi)\big);$$

then for any weight vector $w$ in $V$,

$$\big(w \otimes v_0', \Phi(z)v_0\big) = z^{-(\mu|v\pi)}\big(w \otimes v_0', \Phi[0]v_0\big), \tag{4.21}$$

and

$$\zeta\Phi(z) = (k+g)z\frac{\mathrm{d}}{\mathrm{d}z}\Phi(z) + {:}L_a \otimes J_a\ \Phi(z){:} = 0. \tag{4.22}$$

This is the “classical” Knizhnik-Zamolodchikov equation [KZ].

*Alternative Polarizations.*

The polarization defined by (4.17), (4.18) is *ad hoc*. We have the freedom of shifting any finite set of summands from $J^+$ to $J^-$, as in

$$J_a = J_a^+ + J_a^- = \sum_{n>0} z^n L_a^{-n} + \sum_{n\geq 0} z^{-n} L_a^n;$$

the effect in this particular case is merely to change the sign of $C(\pi)$ in (4.20). The result now agrees with [FR].

Another polarization is suggested by (4.11),

$$\zeta = L_a \otimes J_a = L_a \otimes J_a^+ + L_a \otimes J_a^- = \hat{r}^{\mathrm{t}} + \hat{r}.$$

Here we are dealing directly with the full Kac-Moody algebra, including the $c, d$-terms in $\hat{r}$. Formally, the intertwining property gives

$$\hat{r}\Phi = (\hat{r}^i \otimes \hat{r}_i)\Phi = -(\hat{r}^i\hat{r}_i \otimes 1)\Phi + (\hat{r}^i \otimes 1)\Phi\hat{r}_i,$$

and

$$\hat{r}^i\hat{r}_i = \tfrac{1}{2}\big(\textstyle\sum C^n + c \otimes d + d \otimes c\big) + \tfrac{1}{2}[\hat{r}^i, \hat{r}_i]. \tag{4.23}$$

The first term on the right hand side of this last equation, though meaningless, looks like it may be a scalar, and thus ignorable. Proceeding heuristically up to Proposition 4.2, we begin by dropping this term. The other term is an element $\hat{H}$ of the Cartan subalgebra of $\hat{\mathfrak{g}}$,

$$\hat{H} = \tfrac{1}{2}[\hat{r}_i, \hat{r}^i]; \tag{4.24}$$

it is determined up to an additive central element by

$$[\hat{H}, e_\alpha] = [e_\alpha, \hat{r}^i\, \hat{r}_i] = \mathrm{m}[e_\alpha \otimes 1 + 1 \otimes e_\alpha, \hat{r}] = \mathrm{m}\big(\varphi(\alpha, .) \wedge e_\alpha\big) = \varphi(\alpha, \alpha) e_\alpha. \tag{4.25}$$

If we restrict the relation (4.25) to the real simple roots, then it determines a unique element in the Cartan subalgebra of $\mathfrak{g}$, namely

$$H = \tfrac{1}{2} \sum_{\alpha > 0} [E_\alpha, E_{-\alpha}].$$

Therefore, there is a unique element in the extended Cartan subalgebra, of the form

$$\hat{H} = H + gd, \tag{4.26}$$

such that (4.25) holds for the affine root $e_0$ as well. The integer $g$ is the dual Coxeter number of $\mathfrak{g}$. The redefined operator is

$$\begin{aligned} \zeta\Phi(z) &:= (\hat{H} \otimes 1)\Phi(z) + {:}(\hat{r}^{\mathrm{t}} + \hat{r})\Phi(z){:} \\ &= (\hat{H} \otimes 1)\Phi(z) + \hat{r}^{\mathrm{t}}(z)\Phi(z) + (\hat{r}^i(z) \otimes 1)\Phi(z)\hat{r}_i. \end{aligned} \tag{4.27}$$

Notice that we did not actually use (4.24); instead we defined $\hat{H}$ as the element of $\hat{\mathfrak{g}}$ that has the same commutator with $e_\alpha$ as (4.23). This makes it plausible that the term that was dropped is a scalar and that covariance is preserved. Indeed we have the

**Proposition 4.2.** *The operator product $\zeta\Phi(z)$ defined in (4.27) is covariant; that is, if $\Phi$ is an intertwiner then so is $\zeta\Phi$.*

*Proof.* The coproduct is that of the classical limit, $\Delta(x) = x \otimes 1 + 1 \otimes x$ for $x \in \hat{\mathfrak{g}}$.

$$\begin{aligned} \Delta(x)\zeta\Phi(z) - \big(\zeta\Phi(z)\big)x = &([x, \hat{H}] \otimes 1)\Phi(z) + [\Delta(x), \hat{r}^{\mathrm{t}}(z)]\Phi(z) \\ &+ ([x, \hat{r}^i(z)] \otimes 1)\Phi(z)\hat{r}_i + (\hat{r}^i(z) \otimes 1)\Phi[x, \hat{r}_i\,]. \end{aligned} \tag{4.28}$$

Suppose first that $x \in \mathfrak{g}$; then in terms 2, 3, 4 only the zero modes contribute. The sums over the degree are now finite and

$$\Delta(x)\zeta\Phi(z) - \zeta\Phi(z)x = ([x, H] \otimes 1)\Phi(z) + ([x, r^i(0) r_i(0)] \otimes 1)\Phi(z), \tag{4.29}$$

which vanishes in view of (4.26). We shall verify that (4.28) holds for $x = e_0$. Besides (4.29) there are additional terms that arise from the extension term in the commutation relations, others that arise from the fact that $e_0$ does not commute with the degree operator, and finally the more subtle contributions that come from the fact that the degree of $[e_0, y]$ is shifted by 1 from that of $y$:

$$
\begin{aligned}
([e_0,\hat H]\otimes 1)\Phi &= (-ge_0\otimes 1)\Phi + ([e_0,H]\otimes 1)\Phi,\\
[\Delta(e_0),\hat r^{\mathrm t}(z)]\Phi &= [\Delta(e_0),\hat r^{\mathrm t}(0)]\Phi + \sum_{n>0}[\Delta(e_0),L_a^n\otimes L_a^{-n}]\Phi\\
&= [\Delta(e_0),\hat r^{\mathrm t}(0)]\Phi + (L_a^1\otimes[e_0,L_a^{-1}])\Phi,\\
([e_0,\hat r^i(z)]\otimes 1)\Phi\hat r_i &= ([e_0,\hat r^i(0)]\otimes 1)\Phi\hat r_i + \sum_{n>0}([e_0,L_a^{-n}]\otimes 1)\Phi L_a^n,\\
(\hat r^i(z)\otimes 1)\Phi[e_0,\hat r_i\,] &= (\hat r^i(0)\otimes 1)\Phi[e_0,\hat r_i\,] + \sum_{n>0}(L_a^{-n}\otimes 1)\Phi[e_0,L_a^n].
\end{aligned}
$$

The two infinite sums almost cancel, leaving only the first term of the second one. The sum of the last two expressions is

$$
[\Delta(e_0),\hat r(0)]\Phi + ([e_0,r^i(0)r_i(0)]\otimes 1)\Phi + ([e_0,L_a^{-1}]\otimes 1)\Phi L_a^1.
$$

Adding the second expression we obtain

$$
\begin{aligned}
&[\Delta(e_0),\hat r(0)+\hat r^{\mathrm t}(0)]\Phi + ([e_0,L_a^{-1}]\otimes L_a^1)\Phi + (L_a^1\otimes[e_0,L_a^{-1}])\Phi\\
&\qquad + ([e_0,r^i(0)r_i(0)]\otimes 1)\Phi + ([e_0,L_a^{-1}]L_a^1\otimes 1)\Phi.
\end{aligned}
$$

The first three terms cancel exactly and we have

$$
\begin{aligned}
\Delta(e_0)\zeta\Phi(z) - \zeta\Phi(z)e_0 &= -g(e_0\otimes 1)\Phi + ([e_0,H]\otimes 1)\Phi\\
&\quad + ([e_0,r^i(0)r_i(0)]\otimes 1)\Phi + ([e_0,L_a^{-1}]L_a^1\otimes 1)\Phi.
\end{aligned}
$$

Terms two and three cancel as in (4.29) and the proposition is proved when we verify that, in the evaluation module, $[e_0,L_a^{-1}]L_a^1 = [e_0,L_a]L_a = ge_0$, and repeat the calculation with $e_0$ replaced by $e_{-0}$.

*Normalization.*

Returning to (4.27) we put the degree operator into evidence:

$$
\zeta\Phi(z) = (k+g)z\frac{\mathrm d}{\mathrm dz}\Phi + (H\otimes 1)\Phi(z) + r^{\mathrm t}(z)\Phi(z) + \big(r^i(z)\otimes 1\big)\Phi(z)r_i. \tag{4.30}
$$

Again we fix the grading of the intertwiner as in (4.21), but now with $(\mu|\nu,\pi)$ replaced by

$$
(\mu|\nu) = \frac{A(w)}{k+g},\quad A := \varphi(.,v_0) + \varphi(v_0',.) + H = \tfrac12\big(C(\mu)-C(\nu)\big), \tag{4.31}
$$

so that the operator form of the Knizhnik-Zamolodchikov equation takes the form

$$
\zeta\Phi(z) = 0. \tag{4.32}
$$

Note that this makes the grading of $\Phi$ independent of the choice of evaluation module; this grading/normalization is thus “universal”.

*Remark 4.3.* In view of the interpretation of the quantum field $\Phi(z)$ as an intertwiner for highest weight affine Kac-Moody modules, the appearance of the *rational* r-matrix in the original KZ

equation (4.22) has always seemed somewhat mysterious. The mystery is deepened by the discovery [K] that the monodromy associated with the solutions yields a representation of $U_q(\mathfrak{g})$. The alternative, to use the polarization based on the decompostion $\zeta = \hat{r}+\hat{r}^{\rm t}$, was first suggested in [FR]; it seems to be more natural. However, $\Phi$ is defined as an intertwiner of Kac-Moody modules, with the classical coproduct; it knows nothing about r-matrices. Normal ordering is an example of additive renormalization, or "subtraction", necessary only if the ordinary product is ill defined. Any two polarizations that eliminate the divergent term by subtracting a scalar (that is; without compromising covariance) are equivalent, and one is not more natural than the other, in the present context at least. The fact that renormalization is required is revealed by the fact that the subtracted term, the last term in (4.15), is divergent. It is related to the fact that the classical r-matrix has a pole at $z_1/z_2 = 1$.

*Remark 4.4.* The appearance of the factor $k+g$ as a coefficient of the degree operator in both versions is justified by covariance, as is indeed implied by the proof of Proposition 4.1 in [FR]. The term $(H\otimes 1)\Phi(z)$ has exactly the same origin. Perhaps it should be pointed out that the concept of "covariance" that is evoked in this Section is quite distinct from the covariance under twisting that is alluded to in the title of the paper and in from Section 6 forward.

## 5. The quantum KZ equation.

Here we shall make sense of Eq.(4.2),

$$(Z - \langle Z\rangle)\Phi(z) = 0, \quad Z = RR^{\rm t},$$

in the quantized Kac-Moody algebra, to recover the q-KZ equation of Frenkel and Reshetikhin [FR].

The action of $R^{\rm t}\Phi$ on $V_{\mu,k}$ is well defined (in terms of formal series), since both $\Phi$ and $R^{\rm t}$ act by degree-decreasing operators in the second space (Proposition 3.1 and Eq.(3.9)), but the subsequent action of $R$ is not. We therefore investigate the effect of normal ordering. Thus if

$$R = R^i\otimes R_i,$$

we set (tentatively)

$$:Z\Phi: = (R^i\otimes 1)\Psi R_i, \quad \Psi := R^{\rm t}\Phi \quad (?),$$

and try to prove that the operator $Z : \Phi \mapsto :Z\Phi:$ is invariant; that is, that it commutes with the coproduct. In view of the intertwining property of $\Phi$ this is the same as

$$\Delta(x):Z\Phi: = :Z\Phi:x \quad (?).$$

Attempts to verify this equation leads to

**Proposition 5.1.** *Let $\hat{H}$ be the element in the Cartan subalgebra $\mathcal{A}'_0$ of $\mathcal{A}'$ with the property*

$$q^{\hat{H}}e_\alpha q^{-\hat{H}} = q^{\varphi(\alpha,\alpha)}e_\alpha, \tag{5.1}$$

*and define the normal-ordered product $:Z\Phi:$ by*

$$:Z\Phi: := (\hat{R}^i\otimes 1)\Psi R_i, \quad \Psi := R^{\rm t}\Phi, \quad \hat{R}^i := R^i q^{\hat{H}}. \tag{5.2}$$

*Then*

$$\Delta(x){:}Z\Phi{:} = {:}Z\Phi{:}x, \quad \forall x \in \mathcal{A}'. \tag{5.3}$$

*Proof.* We begin with $R\,\Delta'(e_\alpha) = \Delta(e_\alpha)R$; that is

$$(R^i \otimes R_i)(e_\alpha \otimes 1 + q^{\varphi(\alpha,.)} \otimes e_\alpha) = (1 \otimes e_\alpha + e_\alpha \otimes q^{\varphi(\alpha,.)})(R^i \otimes R_i),$$

and thus

$$R^i \otimes R_i e_\alpha = -R^i e_\alpha q^{-\varphi(\alpha,.)} \otimes R_i + R^i q^{-\varphi(\alpha,.)} \otimes e_\alpha R_i + e_\alpha R^i q^{-\varphi(\alpha,.)} \otimes q^{\varphi(\alpha,.)} R_i\,,$$

which gives us

$$\begin{aligned} {:}Z\Phi{:}e_\alpha = (R^i q^{\hat{H}} \otimes 1)\Psi R_i e_\alpha &= -(R^i e_\alpha q^{-\varphi(\alpha,.)} q^{\hat{H}} \otimes 1)\Psi R_i \\ &+ (R^i q^{-\varphi(\alpha,.)} q^{\hat{H}} \otimes 1)\Psi e_\alpha R_i + (e_\alpha R^i q^{-\varphi(\alpha,.)} q^{\hat{H}} \otimes 1)\Psi q^{\varphi(\alpha,.)} R_i\,. \end{aligned}$$

Using the intertwining property of $\Psi$ we convert the last two terms to

$$(R^i q^{\hat{H}} \otimes e_\alpha)\Psi R_i + (R^i q^{-\varphi(\alpha,.)} q^{\hat{H}} e_\alpha \otimes 1)\Psi R_i + (e_\alpha R^i q^{\hat{H}} \otimes q^{\varphi(\alpha,.)})\Psi R_i.$$

As for the first term, we shift the operator $e_\alpha$ to the right; since $e_\alpha$ commutes with $\hat{H} - \varphi(\alpha,.)$ we get the required cancellation and the result is

$${:}Z\Phi{:}e_\alpha = (\hat{R}^i \otimes e_\alpha)\Psi R_i + (e_\alpha \hat{R}^i \otimes q^{\varphi(\alpha,.)})\Psi R_i = \Delta(e_\alpha)(\hat{R}^i \otimes 1)\Psi R_i.$$

In the classical limit the $q$-factor in (5.2) produces the $\hat{H}$-term in Eq.(4.27). A similar calculation with $e_{-\alpha}$ completes the proof of Proposition 5.1, and we have an independent confirmation of the covariance of (4.27).

*Normalization.*

Our next task is to pull out the degree operator. Since the first space is an evaluation module, on which the central element $c$ is zero, the degree operator appears only in the first factors of $R$ and $R^{\mathrm{t}}$, as $z\frac{\mathrm{d}}{\mathrm{d}z}$. We define $L^{\mp}$ by

$$\hat{R}^i(z) \otimes R_i = q^{(1-u)kd+gd}\big(\mathrm{Ad}(q^{-gd}) \otimes 1\big)L^-(z), \quad R^{\mathrm{t}}(z) = q^{ukd}L^+(z), \tag{5.4}$$

where $d = z\frac{\mathrm{d}}{\mathrm{d}z}$ acts in the evaluation module and $\mathrm{Ad}(x)y = xyx^{-1}$. Objects denoted by the letter $L$ (with ornamentation) do not contain $d$. We also need the expansions

$$L^-(z) = L^{-i}(z) \otimes L_i^-\,, \quad L^+(z) = L^{+i}(z) \otimes L_i^+.$$

Now

$$\begin{aligned} {:}Z\Phi(z){:} &= q^{(1-u)kd+gd}\big(\mathrm{Ad}(q^{-gd})L^{-i}(z) \otimes 1\big)(q^{ukd} \otimes 1)L^+(z)\Phi(z)L_i^- \\ &= q^{(k+g)d}\big(L^{-i}(q^{-g-uk}z) \otimes 1\big)L^+(z)\Phi(z)L_i^- \\ &=: q^{(k+g)d}{:}L(z)\Phi(z){:}. \end{aligned} \tag{5.5}$$

Thus, the q-KZ equation for $\Phi$:

$$\Phi(q^{-k-g}z) = {:}L(z)\Phi(z){:} = \big(L^{-i}(q^{-g-uk}z)\otimes 1\big)L^+(z)\Phi(z)L_i^-\,. \tag{5.6}$$

Here we have fixed $\langle Z\rangle = 1$. It means that the grading of $\Phi$ is so chosen that, on any weight vector $w \in V$ and the highest weight vectors $v_0 \in V_{\mu,k}, v_0' \in V_{\nu,k}$, we have

$$\frac{(w\otimes v_0', \Phi(q^{-k-g}z)v_0)}{(w\otimes v_0', \Phi(z)v_0)} = q^{(\mu|\nu)},$$

with $(\mu|\nu)$ as in (4.31).

The other intertwiner, $\Psi \propto R^{\rm t}\Phi$, satisfies $\Psi x = \Delta'(x)\Psi$ and ${:}Z'\Psi{:} = \Psi$ where $Z' := R^{\rm t}R$. We find

$${:}Z'\Psi(z){:} = R^{\rm t}(\hat R^i\otimes 1)\Psi(z)R_i = q^{ukd}L^+(z)q^{(k-uk+g)d}\big(L^{-i}(q^{-g}z)\otimes 1\big)\Psi(z)L_i^-\,, \tag{5.7}$$

and thus, the q-KZ equation for $\Psi$:

$$\Psi(q^{-k-g}z) = {:}L'(z)\Psi(z){:} = L^+(q^{-g-k+uk}z)\big(L^{-i}(q^{-g}z)\otimes 1\big)\Psi(z)L_i^-\,. \tag{5.8}$$

# 6. Hopf twisting.

It is remarkable that the elliptic quantum group can be viewed as deformation of the trigonometric quantum group. The deformation does not affect the algebraic structure, which remains that of a quantized, affine Kac-Moody algebra. Only the coproduct distinguishes the elliptic case from the trigonometric one. The deformation is implemented by a twist in the category of Hopf algebras (this section) or quasi Hopf algebras (next section). The full elliptic quantum group is quasi Hopf; it becomes Hopf on the quotient by the ideal generated by the center. In this Section we investigate the effect of twisting on the intertwiners and on the KZ equation, in the quantum case where the relationship between the intertwiner and the R-matrix is more clear.

**Definition 6.1.** A formal Hopf deformation of a standard R-matrix $R$ is a formal power series

$$R_\epsilon = R + \epsilon R_1 + ...\ ,$$

that satisfies the Yang-Baxter relation to each order in $\epsilon$.

It turns out [Fr1] that the deformations of greatest interest have the form of a twist.

**Theorem 6.2.** *Let $R$ be the R-matrix, $\Delta$ the coproduct, of a coboundary Hopf algebra $\mathcal{A}'$, and $F\in\mathcal{A}'\otimes\mathcal{A}'$, invertible, such that*

$$\big((1\otimes\Delta_{21})F\big)F_{12} = \big((\Delta_{13}\otimes 1)F\big)F_{31}. \tag{6.1}$$

*Then*

$$\tilde R := (F^{\rm t})^{-1}RF$$

*(a) satisfies the Yang-Baxter relation and (b) defines a Hopf algebra $\tilde{\mathcal{A}}$ with the same product and with coproduct*

$$\tilde{\Delta} = (F^{\rm t})^{-1}\Delta F^{\rm t}.$$

This is a result of Drinfel'd [D2]; a detailed proof was given in [Fr1].

We say that a deformation $R_\epsilon$ of a standard R-matrix $R$ is implemented by a *twistor* $F_\epsilon$ if there is a formal power series

$$F_\epsilon = 1 + \epsilon F_1 + ...$$

that satisfies (6.1) to each order in $\epsilon$ and

$$R_\epsilon = (F_\epsilon^{\rm t})^{-1} R F_\epsilon. \tag{6.2}$$

In this case the deformed R-matrix intertwines a deformed coproduct,

$$R_\epsilon \Delta'_\epsilon = \Delta_\epsilon R_\epsilon, \quad \Delta'_\epsilon := F_\epsilon^{-1}\Delta' F_\epsilon. \tag{6.3}$$

Known solutions of (6.1) have the following structure [Fr1]. We need a pair of subalgebras $\Gamma_1, \Gamma_2$ of $\mathcal{A}' = \hat{\mathfrak{g}}$, generated by sets $\hat{\Gamma}_i \subset \{e_\alpha\}_{\alpha\in N}$, and a diagram isomorphism $\tau : \hat{\Gamma}_1 \to \hat{\Gamma}_2$. A deformation exists when the parameters of $\mathcal{A}'$ satisfy the following condition,

$$\varphi(\sigma,.) + \varphi(.,\tau\sigma) = 0, \quad \sigma \in \hat{\Gamma}_1.$$

Note that $e_{\tau\sigma}$ is defined only if $e_\sigma \in \hat{\Gamma}_1$. Then there is a cocycle $F_\epsilon$ of the form

$$\begin{gathered} F_\epsilon = \prod_{m\geq 1} F_\epsilon^m := F_\epsilon^1 F_\epsilon^2 ... F_\epsilon^m ..., \quad F_\epsilon^m = \sum_{(\sigma)} \epsilon^{mn} F_{(\sigma)}^{m(\rho)} f_{\sigma_1}...f_{\sigma_n} \otimes f_{-\rho_1}...f_{-\rho_n}, \\ f_\sigma := q^{-\varphi(\sigma,.)} e_\sigma, \quad f_{-\rho} := e_{-\rho}\, q^{\varphi(.,\rho)}, \end{gathered} \tag{6.4}$$

where the sum is over all $(\sigma) = \sigma_1, ..., \sigma_n$, and all permutations $(\sigma')$ of $(\sigma)$, such that $\rho_i = \tau^m \sigma'_i$ is defined. We take $F_{(\sigma)}^{m(\rho)} = 1$ when the set $(\sigma)$ is empty.

Note that the family of deformation of this type is large enough to contain the quantization of all the classical Lie bialgebras classified by Belavin and Drinfel'd, with r-matrices of constant, trigonometric and elliptic type. Two cases need to be distinguished.

(a) *Finite twisting* is by definition the case when there is $k$ such that for all $\sigma$, $\tau^k\sigma \notin \hat{\Gamma}_1$; then $\hat{\Gamma}_1, \hat{\Gamma}_2$ are distinct and the product over $m$ is finite.

(b) *Elliptic twisting.* The only other possibility (see [Fr1], Section 16) is that $\mathcal{A}' = \widehat{\mathfrak{sl}(N)}$ and $\Gamma_1 = \Gamma_2$ is generated by all the simple roots. This section deals with twisting in the category of Hopf algebras; elliptic twisting within the context of Hopf algebras implies [Fr2] that we drop the central extension and descend to loop algebras. The full elliptic Kac-Moody algebra is quasi Hopf and will be discussed in the next section.

The deformed R-matrix and coproduct are

$$R_\epsilon = (F_\epsilon^{\rm t})^{-1} R F_\epsilon, \quad \Delta_\epsilon = (F_\epsilon^{\rm t})^{-1}\Delta F_\epsilon^{\rm t}. \tag{6.5}$$

Here are some products that seem ill defined; thus $R$ has degree-increasing operators in the second space, where $F_\epsilon$ has degree-decreasing operators. This problem can be handled in a

general way by adopting an interpretation that is quite natural in deformation theory. One notes that $F_\epsilon$ is a formal power series in the deformation parameter $\epsilon$. One interprets all the operators this way; then the problem reduces to making sure that the coefficients are well defined. Indeed, to any fixed order in $\epsilon$, the product $RF_\epsilon$ is, in the second space, a power series in the operators $e_\alpha$ multiplied by a polynomial in the other generators.

It is, nevertheless, of some interest to determine whether singularities arise as one assigns a value to $\epsilon$ and attempts to sum up the deformation series. In this respect cases (a) and (b) are quite different.

(a) Finite twisting. The sum in (6.4) becomes finite when projected on a finite dimensional representation in either one of the two spaces. Infinite sums will appear if both representations are infinite, but there is a finite number of terms with fixed weight; therefore no infinite, purely numerical series will appear. Infinite sums with operator coefficients are beyond (our power of) analysis in the general case, and of no immediate concern to us. The value of $\epsilon$ is basis dependent; the only distinct possibilities are $\epsilon = 0, 1$.

(b) We note that the range of $\epsilon$ is in this case $|\epsilon| < 1$. Here the situation is more delicate, and of some interest. Under twisting, the Casimir element $Z$ suffers an equivalence transformation

$$Z_\epsilon = (F_\epsilon^{\rm t})^{-1} Z F_\epsilon^{\rm t}, \tag{6.6}$$

and one expects that an intertwiner $\Phi_\epsilon$, satisfying

$$\Delta_\epsilon(x)\Phi_\epsilon = \Phi_\epsilon x, \quad x \in \mathcal{A}', \tag{6.7}$$

may be expressed as $\Phi_\epsilon = (F_\epsilon^{\rm t})^{-1}\Phi$. However, $F_\epsilon^{\rm t}$ has a structure similar to that of $R$, with degree-increasing operators in the second space, and we must consider the possibility that normal ordering may be required. In fact probably not, but since we have not proved this, we shall switch our attention to the other intertwiner.

We consider instead the alternative intertwiner $\Psi$, and the alternative Casimir operator $Z'$ that commutes with $\Delta'(x)$, namely

$$Z' = R^{\rm t} R, \quad Z'\Delta'(x) = \Delta'(x) Z', \quad (Z' - 1)\Psi = 0. \tag{6.8}$$

We have

$$Z'_\epsilon = F_\epsilon^{-1} Z' F_\epsilon, \quad \Psi_\epsilon = F_\epsilon^{-1}\Psi. \tag{6.9}$$

The operator product $F_\epsilon^{-1}\Psi$ is well defined as an operator on $V_{\mu,k}$. The intertwining property of $\Psi_\epsilon$, namely

$$\Delta'_\epsilon(x)\Psi_\epsilon = \Psi_\epsilon x, \tag{6.10}$$

is therefore in the clear.

We define the operator product $Z'_\epsilon\Psi_\epsilon(z)$. Formally,

$$Z'_\epsilon\Psi_\epsilon = R_\epsilon^{\rm t} R_\epsilon \Psi_\epsilon = F_\epsilon^{-1} Z'\Psi,$$

and this too is well defined, provided we define the untwisted product as in (5.7); that is

$$Z'\Psi \to {:}Z'\Psi{:} = R^{\rm t}(R^i q^{\hat H} \otimes 1)\Psi R_i.$$

The equation satisfied by the twisted correlation function is $(Z'_\epsilon - 1)\Psi_\epsilon = 0$ or more precisely

**Definition 6.3.** The twisted q-KZ equation is the following equation for the twisted intertwiner operator,

$$\Psi_\epsilon(q^{-k-g}z) = F_\epsilon^{-1}(q^{-k-g}z)L^+(zq^{-g-k+uk})\big(L^{-i}(zq^{-g})\otimes 1\big)F_\epsilon(z)\Psi_\epsilon(z)L_i^-. \tag{6.11}$$

It should be noted that the polarization used is the same as before deformation. To justify this we repeat that the definition of the intertwining operators is independent of normal ordering conventions, normal ordering is relevant only when the ordinary product does not exist, it is required to be well defined and covariant, nothing more. Of course, it is also true that, if $\Psi_\epsilon$ is defined as in (6.9), then (6.11) is equivalent to (5.8).

The top matrix element of $\Psi_\epsilon$ is

$$(v_0', \Psi_\epsilon v_0) = (v_0', F_\epsilon^{-1}\Psi v_0) = (v_0', \Psi v_0). \tag{6.12}$$

This shows that, in a complete description of, say, the eight-vertex model, both periodic and non-periodic functions appear. We had naively expected to encounter nothing but elliptic functions, that "the eight-vertex model lives on the torus".

Having thus discarded a prejudice, we are comfortable with the continued use, in the twisted case, of the original polarization based on the standard trigonometric R-matrix. The alternative of defining a normal-ordered product such that

$$R_\epsilon\Phi_\epsilon = (R_\epsilon^i\otimes 1)\Phi_\epsilon R_{\epsilon i}$$

is entirely redundant.

Another idea is to replace matrix elements by traces, as suggested by Bernard [Ber] and in [FR]. However, since we know that the elliptic quantum group, as an algebra, is the same as the standard quantum group (that is, a Kac-Moody algebra), there seems to be no reason to take less interest in the highest weight matrix elements in the elliptic case. Continuity of physics also suggests that we continue to work with the usual module structure, as was argued in [JMN], Section 4. Trace functionals are interesting in themselves, but there seems to be no reason to neglect the matrix elements.

The intertwiners of Kac-Moody modules, and the solutions of the KZ equation, know nothing about r-matrices. For all that we may derive different versions of the equation, the solutions remain the same. To base the polarization on the R-matrix is not an imperative; more important is to adopt a workable definition that gives a meaning to the objects of interest; to wit, matrix elements of intertwiners.

In the setting of conformal field theory twisting does not affect the quantization paradigm, but it does change the quantum fields (the intertwiners) and their operator product expansions.

We shall need to know the twistor $F_\epsilon$. It is determined, uniquely, by the recursion relations [Fr1]

$$[1\otimes f_\rho, F_\epsilon^m] = \epsilon^m\big(F_\epsilon^m(f_{\tau^{-m}\rho}\otimes q^{-\varphi(\rho,.)}) - (f_{\tau^{-m}\rho}\otimes q^{\varphi(.,\rho)})F_\epsilon^m\big), \tag{6.13}$$

with the initial conditions

$$F_\epsilon^m = 1 - \epsilon^m\sum_{(\rho)} f_{\tau^{-m}\rho}\otimes f_{-\rho} + \dots . \tag{6.14}$$

These equation were solved in a special case, and used to calculate the elliptic R-matrix of $\widehat{\mathfrak{sl}(2)}$ in the fundamental representation [Fr1]. Later, we shall exploit the similarity between this relation and the recursion relation (2.4) for the universal R-matrix.

# 7. Quasi Hopf twisting.

We are interested in the elliptic quantum groups, in the sense of Baxter [Ba] and Belavin [Be]. This takes us out of the framework of Hopf algebras, but just barely so. The special nature of these quasi Hopf algebras is that they become Hopf algebras at level zero; that is, on the quotient by the ideal generated by the center.

Quasi Hopf deformations are constructed in the same way as Hopf deformations, except that the element $F_\epsilon$ need not satisfy the cocycle condition (6.1). The deformed R-matrix and coproduct are given by (6.5), but the former no longer satifies the Yang-Baxter relation and the latter is not coassociative, in general.

If $F^{m(\rho)}_{(\sigma)}$ are the coefficients of the elliptic Hopf twistor in (6.4), then the elliptic quasi Hopf twistor has the form [Fr2]

$$F_\epsilon = \prod_{m=1,2,...} F^m_\epsilon, \quad F^m_\epsilon = \sum_{(\sigma)} \epsilon^{nm} F^{m(\rho)}_{(\sigma)} f_{\sigma_1}...f_{\sigma_n} \otimes f_{-\rho_1}...f_{-\rho_n} Q(m,\rho), \tag{7.1}$$

where $Q(m,\rho) \in \mathcal{A}'_0 \otimes \mathcal{A}'_0$ and $\mathcal{A}'_0$ is the Cartan subalgebra of the quantized Kac-Moody algebra $\mathcal{A}'$. This factor is equal to unity in the Hopf case, and (7.1) then reduces to (6.4). The $F_\epsilon$-twisted algebra is a Hopf algebra when the parameters satisfy the condition

$$\varphi(\sigma,.) + \varphi(.,\tau\sigma) = 0, \quad \sigma \in \hat{\Gamma}_1,$$

where now $\tau$ is the cyclic diagram automorphism that takes each simple root of $\mathfrak{sl}(N)$ to its neighbour. This condition can be satisfied on the loop algebra (when $c \mapsto 0$). We are interested in the full Kac-Moody algebra ($c \neq 0$); in that case the best that can be done is to choose parameters such that

$$\varphi(\sigma,.) + \varphi(.,\tau\sigma) = [(1-u)\delta^0_\sigma + u\delta^0_{\tau\sigma}]c. \tag{7.2}$$

This algebra is what we mean by "elliptic quantum group in the sense of Baxter and Belavin"; it is a quasi Hopf algebra of a particularly benevolent type, where the deviation from coassociativity is confined to the center.

Instead of the cocycle condition (6.1) we now have

$$\big((\mathrm{id} \otimes \Delta_{21})F_\epsilon\big)F_{\epsilon 12} = \big((\Delta_{13} \otimes \mathrm{id})F_\epsilon\big)F_{\epsilon 31,2}, \tag{7.3}$$

where $F_{\epsilon ij,k}$ is an extension of $F_{\epsilon ij}$, supported on the center, to the $k$'th space. In the case of interest, when we are dealing with modules with fixed level $c \mapsto k$, this amounts to a modification of the coefficients in $F_{\epsilon ij}$. From (7.3) one gets the Cartan factors $Q(m,\rho)$ [Fr2] and the recursion relation

$$[1 \otimes f_\rho, F^m_\epsilon] = \epsilon^m \big(F^m_\epsilon (f_{\tau^{-m}\rho} \otimes q^{-\varphi(\rho,.)}) - (f_{\tau^{-m}\rho} \otimes q^{\varphi(.,\rho)})F^m_\epsilon\big)Q(m,\rho), \tag{7.4}$$

with the initial conditions

$$F^m_\epsilon = 1 - \epsilon^m \sum_\rho (f_{\tau^{-m}\rho} \otimes f_{-\rho})Q(m,\rho) + ... \,. \tag{7.5}$$

The solutions will be given later. Once $F^m_{\epsilon 12}$ is known, $F^m_{\epsilon 12,3}$ is obtained by means of the substitution

$$1 \otimes c \mapsto 1 \otimes \Delta(c). \tag{7.6}$$

## 8. Correlation Functions.

The main objects of interest, in conformal field theory as well as in the study of statistical models, are the correlation functions. In their simplest form they are matrix elements of products of intertwiners,

$$
\begin{aligned}
f_{v'v}(z_1,...,z_N) &= \langle v', \Phi(z_1)...\Phi(z_N)\, v\rangle \in V_1(z_1)\otimes ... \otimes V_N(z_N),\\
g_{v'v}(z_1,...,z_N) &= \langle v', \Psi(z_1)...\Psi(z_N)\, v\rangle \in V_1(z_1)\otimes ... \otimes V_N(z_N).
\end{aligned}
\tag{8.1}
$$

Here $\Phi(z_p)$ and $\Psi(z_p)$ are intertwiners between highest weight modules,

$$
\Phi(z_p), \Psi(z_p) : V_{\mu_p,k} \to V_p(z_p)\otimes V_{\mu_{p-1},k}, \quad p = 1,...,N,
$$

with $\{V_p(z_p)\}$ a set of evaluation modules, and $v \in V_{\mu_N,k}$, $v' \in V_{\mu_0,k}$. These "functions" are formal, $V_1\otimes ...\otimes V_N$-valued series in $N$ distinct variables.

*Classical Correlation Functions.*

We begin with the classical case and the polarization (4.18),

$$
J_a = J_a^+ + J_a^- = \sum_{n>0} z^n L_a^{-n} + \sum_{n\geq 0} z^{-n} L_a^n,
$$

and the normalization that leads to (4.22):

$$
(k+g) z\frac{\mathrm{d}}{\mathrm{d}z}\Phi(z) + L_a\otimes J_a^+\Phi(z) + (L_a\otimes 1)\Phi(z) J_a^- = 0.
$$

Then for any $p\in\{1,...,N\}$,

$$
\begin{aligned}
&(k+g) z_p\frac{\mathrm{d}}{\mathrm{d}z_p} f_{v'v}(z_1,...,z_N) =\\
&- L_a^{(p)}\langle v', ...\Phi(z_{p-1})J_a^+(z_p)\Phi(z_p)...v\rangle - L_a^{(p)}\langle v', ...\Phi(z_{p-1})\Phi(z_p)J_a^-(z_p)...v\rangle.
\end{aligned}
\tag{8.2}
$$

Here $L_a^{(p)}$ denotes the action of $L_a$ in $V_p$. Suppose now that the vectors $v_0$ and $v_0'$ are highest weight vectors of the respective highest weight modules. The intertwiners satisfy $[L_a^n, \Phi(z_p)] = -L_a^{(p)} z_p^n \Phi(z_p)$; this allows us to permute $J^+$ through to the left, where it dies on the highest weight vector, and to permute $J^-$ towards the right, where only the zero modes survive, to contribute the last term in

$$
\begin{aligned}
(k+g) z_p\frac{\mathrm{d}}{\mathrm{d}z_p} f_{v_0'v_0}(z_1,...,z_N) &= -\sum_{1\leq q<p}\,\sum_{n>0}\Big(\frac{z_p}{z_q}\Big)^n L_a^{(q)} L_a^{(p)}\langle v_0', ...\Phi(z_{p-1})\Phi(z_p)...v_0\rangle\\
&+ \sum_{N\geq q>p}\,\sum_{n\geq 0}\Big(\frac{z_q}{z_p}\Big)^n L_a^{(p)} L_a^{(q)}\langle v_0', ...\Phi(z_{p-1})\Phi(z_p)...v_0\rangle + L_a^{(p)}\langle v', ...\Phi(z_{p-1})\Phi(z_p)...L_a v_0\rangle.
\end{aligned}
$$

Hence

$$
(k+g)\frac{\mathrm{d}}{\mathrm{d}z_p} f_{v_0'v_0}(z_1,...,z_N) = \sum_{q\neq p}\frac{1}{z_p - z_q} L_a^{(p)} L_a^{(q)} f_{v_0'v_0}(z_1,...z_N), \quad p = 1,...,N,
\tag{8.3}
$$

where $q$ takes the values $1,...,N+1$, $z_{N+1}=0$, and $L_a^{(N+1)}$ acts on $v_0$. The last expression must be supplemented by the instruction

$$\frac{1}{z_p-z_q} := \begin{cases} (1/z_p)\sum_{n\geq 0}(z_q/z_p)^n, & q>p, \\ (-1/z_q)\sum_{n\geq 0}(z_p/z_q)^n, & q<p. \end{cases} \tag{8.4}$$

The domain of convergence is thus $|z_1|>|z_2|>...>|z_{N+1}|=0$.

In the simplest, nontrivial case $N=1$. Projecting on a vector $w\in V$ we get

$$(k+g)\frac{\mathrm{d}}{\mathrm{d}z_1}f^w_{v_0'v_0}(z_1)=\frac{c}{z}f^w_{v_0'v_0}(z_1), \quad c=\frac{\langle w\otimes v', \mathcal{C}\Phi\, v\rangle}{\langle w\otimes v', \Phi v\rangle}=\tfrac{1}{2}\big(C(\nu)-C(\mu)-C(\pi)\big),$$

which simply reflects the choice of grading of $\Phi$. The case $N=2$ is not much more complicated; the equations are

$$(k+g)\frac{\mathrm{d}f}{\mathrm{d}z_1}=\frac{c_{12}f}{z_1-z_2}+\frac{c_{13}f}{z_1}, \quad (k+g)\frac{\mathrm{d}f}{\mathrm{d}z_2}=\frac{c_{12}f}{z_2-z_1}+\frac{c_{23}f}{z_2}, \tag{8.5}$$

where $c_{ij}=L_a^{(i)}L_a^{(j)}$ and "3" refers to the source space. In the case of $\widehat{\mathfrak{sl}(2)}$ and fundamental evaluation modules it is a simple matter to work out the hypergeometric solutions. The general structure of the solution was exploited by Khono [K] and Drinfel'd [D2] to construct representations of the braid group and examples of quasi Hopf algebras.

If instead we use the polarization of (4.27) we obtain from (4.30) and (4.31), on the vectors of highest weight,

$$\begin{aligned} &(k+g)z_p\frac{\mathrm{d}}{\mathrm{d}z_p}f_{v_0'v_0}(z_1,...,z_N)+\big(A_p+\sum_{q=1}^{p-1}r_{qp}-\sum_{q=p+1}^{N}r_{pq}\big)f_{v_0'v_0}(z_1,...,z_N)=0, \\ &A_p:=(H+\varphi(v_0',.)+\varphi(.,v_0))_p, \end{aligned} \tag{8.6}$$

for $p=1,...,N$. When $N=2$,

$$(k+g)z_1\frac{\mathrm{d}}{\mathrm{d}z_1}f_{v_0'v_0}(z_1,z_2)+A_1f_{v_0'v_0}(z_1,z_2)-r_{12}f_{v_0'v_0}(z_1,z_2)=0,$$

$$(k+g)z_2\frac{\mathrm{d}}{\mathrm{d}z_2}f_{v_0'v_0}(z_1,z_2)+A_2f_{v_0'v_0}(z_1,z_2)+r_{12}f_{v_0'v_0}(z_1,z_2)=0.$$

The solutions are, of course, the same, up to normalization.

*q-Deformed Correlation Functions.*

We turn to the q-KZ equation (5.6),

$$\Phi(q^{-k-g}z)=:L(z)\Phi(z):.$$

For functions of the type (8.1) the implication is

$$\begin{aligned} T_pf_{v_0'v_0}(z_1,...,z_N)&:=f_{v_0'v_0}(...,z_{p-1},q^{-k-g}z_p,z_{p+1},...) \\ &=\langle v_0',...\Phi(z_{p-1})L^{-i}(z_p')L^+(z_p)\Phi(z_p)L_i^-\Phi(z_{p+1})...v_0\rangle, \end{aligned}$$

with $z' = q^{-g-uk}z$. More transparently,

$$T_p f_{v_0' v_0}(z_1,...,z_N) = \big[L^{-i}(z_p')L^{+j}(z_p)\big]\langle v_0', ...\Phi(z_{p-1})L_j^+\Phi(z_p)L_i^-\Phi(z_{p+1})...v_0\rangle. \tag{8.7}$$

For $N=2$,

$$\begin{aligned} f_{v_0' v_0}(q^{-k-g}z_1, z_2) &= \big[L^{-i}(z_1')q^{-\varphi(v_0',.)}\big]_1\langle v_0', \Phi(z_1)L_i^-\Phi(z_2)v_0\rangle,\\ f_{v_0' v_0}(z_1, q^{-k-g}z_2) &= \big[q^{\varphi(.,v_0)+H}L^{+i}(z_2)\big]_2\langle v_0', \Phi(z_1)L_i^+\Phi(z_2)v_0\rangle. \end{aligned} \tag{8.8}$$

We reduce this using the quasi triangularity conditions in the Appendix. The final result is

$$\begin{aligned} T_1 f_{v_0' v_0}(z_1,z_2) &= R_{12}^{-1}(\frac{z_2}{z_1}q^{g+k})q^{A_1}f_{v_0' v_0}(z_1,z_2),\\ T_2 f_{v_0' v_0}(z_1,z_2) &= q^{A_2}R_{12}(\frac{z_2}{z_1})f_{v_0' v_0}(z_1,z_2),\\ A_i &:= \big(\varphi(v_0',.)+\varphi(.,v_0)+H\big)_i, \quad i=1,...,N. \end{aligned} \tag{8.9}$$

These two equations can be combined in two ways. The result is the same in either case, in consequence of the fact that the operator $A_1 + A_2$ (the subscripts refer to the two evaluation modules) commutes with $R_{12}$. From the fact that the correlation function is invariant for the action of the Cartan subalgebra in the four spaces it follows in fact that we can replace

$$A_1 + A_2 \to \tfrac{1}{2}\big(C(\mu) - C(\nu)\big).$$

The result is that

$$T_1T_2 f_{v_0' v_0}(z_1,z_2) = q^{A_1+A_2}f_{v_0' v_0}(z_1,z_2). \tag{8.10}$$

The two equations (8.9) are thus mutually consistent. For correlators with more than two intertwiners one obtains similar equations ([FR] and below), and for them consistency depends on the fact that $R$ satisfies the Yang-Baxter relation.

For the other two-point function we have from (5.8), with $z'' = q^{-g-k+uk}z$,

$$T_p g_{v_0',v_0}(z_1,...,z_N) = L^{+i}(z_p'')L^{-j}(q^{-g}z_p)\langle v_0', ...\Psi(z_{p-1})L_i^+\Psi(z_p)L_j^-\Psi(z_{p+1})...v_0\rangle, \tag{8.11}$$

and, in particular,

$$T_1 g_{v_0',v_0}(z_1,z_2) = L^{+i}(z_1'')L^{-j}(q^{-g}z_1)\langle v_0', L_i^+\Psi(z_1)L_j^-\Psi(z_2)v_0\rangle, \tag{8.12}$$

$$T_2 g_{v_0',v_0}(z_1,z_2) = L^{+i}(z_2'')L^{-j}(q^{-g}z_2)\langle v_0', \Psi(z_1)L_i^+\Psi(z_2)L_j^-v_0\rangle, \tag{8.13}$$

and with the help of the Appendix,

$$\begin{aligned} T_1 g_{v_0',v_0}(z_1,z_2) &= q^{A_1}R_{12}^{-1}(\frac{z_2}{z_1})g_{v_0',v_0}(z_1,z_2),\\ T_2 g_{v_0',v_0}(z_1,z_2) &= R_{12}(\frac{z_2}{z_1}q^{-k-g})q^{A_2}g_{v_0',v_0}(z_1,z_2). \end{aligned} \tag{8.14}$$

The q-KZ equations for the 3-point functions are

$$\begin{aligned} T_1 f(z_1,z_2,z_3) &= R_{12}^{-1}(\frac{z_2}{z_1}q^{k+g})R_{13}^{-1}(\frac{z_3}{z_1}q^{k+g})q^{A_1}f(z_1,z_2,z_3),\\ T_2 f(z_1,z_2,z_3) &= R_{23}^{-1}(\frac{z_3}{z_2}q^{k+g})q^{A_2}R_{12}(\frac{z_2}{z_1})f(z_1,z_2,z_3),\\ T_3 f(z_1,z_2,z_3) &= q^{A_3}R_{13}(\frac{z_3}{z_1})R_{23}(\frac{z_3}{z_2})f(z_1,z_2,z_3), \end{aligned} \tag{8.15}$$

with $A$ as before. Integrability is expressed as a cocycle condition that is precisely the Yang-Baxter relation for $R$, Eq.(A.6) with $c_2 = 0$. (The tilde on $\tilde{R}_{12}$ is redundant.)

*Remarks 8.* (1) It is interesting to note that

$$T_1 T_2 T_3 f(z_1, z_2, z_3) = q^{A_1+A_2+A_3} f(z_1, z_2, z_3) = q^{\frac{1}{2}(C(\mu)-C(\nu))} f(z_1, z_2, z_3). \tag{8.16}$$

This is what one expects, since the product of any number of intertwiners should have the universal property; see Remark 3.2, also (4.31) and (8.10).

(2) The first and the last equations in (8.15) can be written as follows,

$$\begin{aligned} T_1 f(z_1, z_2, z_3) &= q^{-(k+g)d_1} R_{1,32}^{-1} q^{(k+g)d_1} q^{A_1} f(z_1, z_2, z_3), \\ T_3 f(z_1, z_2, z_3) &= q^{A_3} R_{21,3} f(z_1, z_2, z_3). \end{aligned} \tag{8.17}$$

Here $R_{i,jk}$ is the action of the universal R-matrix in the evaluation module via the opposite coproduct, $R_{1,32} = (\mathrm{id} \otimes \Delta')R, R_{21,3} = (\Delta' \otimes \mathrm{id})R$. This too is an expression of universality; compare the first of (8.17) with the first of (8.9).

Similarly one finds directly, using the formulas in the Appendix that, if $g(z_1, z_2, z_3)$ is the alternative 3-point function in (8.1), then

$$\begin{aligned} T_3 g(z_1, z_2, z_3) &= R_{23}(\frac{z_3}{z_2} q^{-k-g}) R_{13}(\frac{z_3}{z_1} q^{-k-g}) q^{A_3} g(z_1, z_2, z_3) \\ &= q^{-(k+g)d_3} R_{12,3} q^{(k+g)d_3} q^{A_3} g(z_1, z_2, z_3). \end{aligned} \tag{8.18}$$

The other two formulas cannot be obtained so directly, but the principle of universality encountered in Remarks 8 tells us that

$$T_1 g(z_1, z_2, z_3) = q^{A_1} R_{1,23}^{-1} g(z_1, z_2, z_3). \tag{8.19}$$

Finally, from (8.23),

$$\begin{aligned} T_2 g(z_1, z_2, z_3) &= T_1^{-1} q^{A_1+A_2+A_3} T_3^{-1} g(z_1, z_2, z_3) \\ &= q^{(k+g)d_1} R_{12} q^{A_2} R_{23}^{-1} q^{-(k+d)d_1} g(z_1, z_2, z_3). \end{aligned} \tag{8.20}$$

Summing up, we have

$$\begin{aligned} T_1 g(z_1, z_2, z_3) &= q^{A_1} R_{1,23}^{-1}\, g(z_1, z_2, z_3), \\ T_2 g(z_1, z_2, z_3) &= q^{-(k+g)d_2} R_{12} q^{(k+g)d_2} q^{A_2} R_{23}^{-1} g(z_1, z_2, z_3), \\ T_3 g(z_1, z_2, z_3) &= q^{-(k+g)d_3} R_{12,3} q^{(k+g)d_3} q^{A_3} g(z_1, z_2, z_3). \end{aligned} \tag{8.21}$$

*Twisting and Covariance.*

Let us evaluate one of the two-point functions of the twisted model,

$$g_\epsilon(z_1, z_2) = \langle v', \Psi_\epsilon(z_1) \Psi_\epsilon(z_2)\, v\rangle = \langle v', F_\epsilon^{-1} \Psi(z_1) F_\epsilon^{-1} \Psi(z_2)\, v\rangle. \tag{8.22}$$

On highest weight vectors,

$$g_\epsilon(z_1,z_2) = \langle v_0', \Psi(z_1)F_\epsilon^{-1}(z_2)\Psi(z_2)\,v_0\rangle = F_\epsilon^{-1,i}(z_2)\langle v_0',\,\Delta'(F_{\epsilon i}^{-1})\Psi(z_1)\Psi(z_2)\,v_0\rangle. \tag{8.23}$$

Now $\Delta'(f_{-\rho}) = f_{-\rho}\otimes 1 + q^{\varphi(.,\rho)}\otimes f_{-\rho}$ and so, for Hopf deformations, when $F_\epsilon$ is a series of the type (6.4),

$$g_\epsilon(z_1,z_2) = F_\epsilon^{-1}(z_2,z_1)g(z_1,z_2). \tag{8.24}$$

An alternative derivation of this result makes direct use of the cocycle condition. It can be written as follows

$$F_{\epsilon 13}^{-1}\big((\mathrm{id}\otimes\Delta_{31})F_\epsilon^{-1}\big) = F_{\epsilon 21}^{-1}\big((\Delta_{12}\otimes\mathrm{id})F_\epsilon^{-1}\big). \tag{8.25}$$

Applying $v_0'$ we get, because this vector is a highest weight vector,

$$\langle v_0', (\mathrm{id}\otimes\Delta_{31})F_\epsilon^{-1})... = \langle v_0', F_{\epsilon 21}^{-1}...\;, \tag{8.26}$$

which is just what we need to reduce (8.23) to (8.24).

The transformation formula (8.24) shows that the result (8.14) is not covariant with respect to twisting, in the following sense. The equation satisfied by $g_\epsilon$ is

$$T_1 g_\epsilon(z_1,z_2) = F_\epsilon^{-1}(z_2,q^{-k-g}z_1)q^{A_1}R_{12}^{-1}(\frac{z_2}{z_1})F_\epsilon(z_2,z_1)g_\epsilon(z_1,z_2);$$

the right hand side is very different from the naive analogue of (8.14),

$$q^{A_1}R_{\epsilon 12}^{-1}(\frac{z_2}{z_1})g_\epsilon(z_1,z_2).$$

Thus twisting does not preserve the form of the equations satisfied by matrix elements of intertwining operators; one cannot simply replace $R$ in these equations by a twisted R-matrix. In fact, it is clear that our calculations made use of the specific form of the standard R-matrix. The factors $q^A$, in particular, are characteristic of the standard R-matrix. Of course, we do not deny the existence of holonomic difference equations that involve R-matrices of a more general type. The claim is that the solutions to such equations are not, in general, matrix elements of intertwining operators for highest weight, quantized Kac-Moody modules. The elliptic correlation functions can be found by solving a "modified" q-KZ equation, but much more simply by the intermediary of the solutions of the standard q-KZ equations for the 6-vertex model, as in Eq.(8.24).

For three-point functions the effect of twisting is

$$\begin{aligned}
g_\epsilon(z_1,z_2,z_3) &= \langle v_0', F_\epsilon^{-1}\Psi(z_1)F_\epsilon^{-1}\Psi(z_2)F_\epsilon^{-1}\Psi(z_3)v_0\rangle\\
&= F_\epsilon^{-1,i}(z_2)F_\epsilon^{-1,j}(z_3)\langle v_0', \Psi(z_1)F_{\epsilon i}^{-1}\Psi(z_2)F_{\epsilon j}^{-1}\Psi(z_3)v_0\rangle\\
&= F_\epsilon^{-1,i}(z_2)F_\epsilon^{-1,j}(z_3)\langle v_0', \Delta_{41}(F_{\epsilon i}^{-1})\Psi(z_1)\Delta_{42}(F_{\epsilon j}^{-1})\Psi(z_2)\Psi(z_3)v_0\rangle\\
&= F_\epsilon^{-1}(z_2,z_1)(F_{\epsilon j}^{-1})_i(z_2)F_\epsilon^{-1,j}(z_3)\langle v_0', \Psi(z_1)(F_{\epsilon j}^{-1})^i\Psi(z_2)\Psi(z_3)v_0\rangle\\
&= F_\epsilon^{-1}(z_2,z_1)(F_{\epsilon j}^{-1})_i(z_2)F_\epsilon^{-1,j}(z_3)\langle v_0', \Delta_{41}((F_{\epsilon j}^{-1})^i)\Psi(z_1)\Psi(z_2)\Psi(z_3)v_0\rangle\\
&= F_\epsilon^{-1}(z_2,z_1)(F_{\epsilon j}^{-1})_i(z_2)F_\epsilon^{-1,j}(z_3)\langle v_0', ((F_{\epsilon j}^{-1})^i)(z_1)\Psi(z_1)\Psi(z_2)\Psi(z_3)v_0\rangle\\
&= F_\epsilon^{-1}(z_2,z_1)\Delta_{12}(F_{\epsilon j}^{-1})(z_1,z_2)F_\epsilon^{-1,j}(z_3)g(z_1,z_2,z_3)\\
&= F_\epsilon^{-1}(z_2,z_1)\big((\mathrm{id}\otimes\Delta)F_\epsilon^{-1}\big)(z_3,z_1,z_2)g(z_1,z_2,z_3).
\end{aligned} \tag{8.27}$$

Thus we conclude that the twisted correlation functions can be obtained from the untwisted ones. The latter are found by solving equations that are known to be integrable by virtue of the fact that the standard R-matrix satisfies the Yang-Baxter relation. It is possible, but redundant and unrewarding, to write down the equations satisfied by the twisted correlation functions; they are complicated and uninstructive whether expressed in terms of $R$ or $R_\epsilon$.

## 9. Correlation Function for the 8-Vertex Model.

Here we try to understand what, if any, are the qualitative new features that result from the fact that the elliptic quantum group is not a Hopf algebra. Technically, the difference is that the reduction of (8.23) to (8.24) is no longer valid, because the twistor is no longer of the type (6.4). Instead of the cocycle condition (6.1) that gave us (8.25) we now have the modified cocycle condition (7.3), which yields

$$F_{\epsilon 13}^{-1}\big((\mathrm{id}\otimes\Delta_{31})F_\epsilon^{-1}\big) = F_{\epsilon 21,3}^{-1}\big((\Delta_{12}\otimes\mathrm{id})F_\epsilon^{-1}\big) \tag{9.1}$$

and, instead of (8.24),

$$g_\epsilon(z_1,z_2) = F_{\epsilon 21,3}^{-1}(z_2,z_1)g(z_1,z_2). \tag{9.2}$$

To calculate $F_{\epsilon 21,3}$ see (7.6). This is the two-point function for the 8-vertex model. The quasi Hopf nature of the elliptic quantum group is parameterized by the level $k$ of the highest weight module and the effect on the two-point function is in the numerical modification of the matrix $F_\epsilon$ that is indicated by the third index. Equation (8.27) gets modified in the same manner.

To get an idea of the importance of this effect it is enough to calculate the modified matrix in the case $N=2$ with $V$ the fundamental $\mathfrak{sl}(2)$-module. The result is as follows. The trigonometric R-matrix is given for comparison, with the two spaces interchanged:

$$R^{\mathrm{t}} = \frac{A(q,x)}{1-q^{-2}x}q^\varphi\Big((1-q^{-2}x)H_+ + (1-x)H_- + e_1\otimes e_{-1} + e_0\otimes e_{-0}\Big), \tag{9.3}$$

$$F_{\epsilon 12,3}^{2m} = \frac{A^{2m}(q,x,\epsilon)}{1-q^2\alpha\alpha' x}\Big((1-q^2\alpha\alpha' x)H_+ + (1-\alpha\alpha' x)H_- - \alpha f_1\otimes f_{-1} - \alpha' f_0\otimes f_{-0}\Big), \tag{9.4}$$

$$F_{\epsilon 12,3}^{2m-1} = \frac{A^{2m-1}(q,x,\epsilon)}{1-q^2\beta^2 x}\Big((1-\beta^2 x)H_+ + (1-q^2\beta^2 x)H_- - \beta f_1\otimes f_{-0} - \beta f_0\otimes f_{-1}\Big). \tag{9.5}$$

Here $x = z_1/z_2$,

$$H_\pm = \tfrac{1}{4}[(1+H)\otimes(1\pm H) + (1-H)\otimes(1\mp H)]$$

(in another notation, $H_+ = e_{11}\otimes e_{11} + e_{22}\otimes e_{22}, H_- = e_{11}\otimes e_{22} + e_{22}\otimes e_{11}$), and

$$\alpha = q^{uk}(\epsilon^2 q^{-k})^m,\ \alpha' = q^{(1-u)k}(\epsilon^2 q^{-k})^m,\ \beta^2 = q^k(\epsilon^2 q^{-k})^{2m-1}.$$

Remember that $k$ denotes the level of the highest weight $\widehat{\mathfrak{sl}(2)}$-module. It enters here because it appears in the extension $F_{\epsilon 21,3}$ of the twistor in Eq.(9.2). This operator acts in three spaces, but its action on the highest weight module is limited to the center. In the level zero case we recover the Hopf twistor. The calculation that leads to (9.3) is given in detail in the Appendix, with an explicit formula for the normalizing factor $A(q,x)$. The matrix factors in (9.4) and (9.5) are obtained in the same way, and the scalar factors as follows.

**Proposition 9.1.** *(a) The normalizing factor in (9.4) is*

$$A^{2m}(q,x,\epsilon) = A(1/q,\alpha\alpha' x). \tag{9.7}$$

*(b) The normalizing factor in (9.5) is*

$$A^{2m-1}(q,x,\epsilon) = A(1/q,\beta^2 x). \tag{9.8}$$

*Proof of (a).* Consider the universal R-matrix, and the algebra map generated by

$$e_1 \to \alpha^{-1} f_1, \quad e_{-1} \to -\alpha f_{-1}, \quad e_0 \to \alpha'^{-1} f_0, \quad e_{-0} \to -\alpha' f_{-0} \tag{9.9}$$

in the second space, but $e_i \to f_i$ in the first space. This maps the original algebra to another algebra with the $q$ replaced by $q^{-1}$. Now consider the factorization (2.1) of the universal R-matrix, $R = q^{\varphi} T$. After (9.9), the first two terms in $T^{\mathrm{t}}$ agree with the first two terms of $F_\epsilon^{2m}$. The recursion relations (2.4) and (7.4) also agree, after replacing $q$ by $1/q$, and so do the solutions. Then we pass to the evaluation representation, setting $f_0 = \hat{z}_1 f_{-1}$. In the R-matrix (more precisely, in $T^{\mathrm{t}}$) we have set $e_0 = z_1 e_{-1}$, which after the substitution (9.9) becomes $\alpha' f_0 = (z_1/\alpha) f_{-1}$, so that $z_1 = \alpha\alpha' \hat{z}_1$. Under these transformations, including transposition of the two spaces, the polynomial factor in (9.3) is transformed into that of (9.4), and the normalizing factor also agree.

*Proof of (b).* In this case, in the second space let $e_1 \to \beta^{-1} f_0, e_{-1} \to -\beta f_{-0}, e_0 \to \beta^{-1} f_1$, $e_{-0} \to -\beta f_{-1}$, and in the first space $e_i \to f_i$.

Putting it all together, we have after a simple change of basis

$$F_{\epsilon 12,3}(z_1, z_2) = A(F_\epsilon) \begin{pmatrix} a & & & \hat{d} \\ & b & \hat{c} & \\ & \hat{c} & b & \\ \hat{d} & & & a \end{pmatrix}, \quad x = z_1/z_2, \tag{9.10}$$

with

$$\begin{aligned} A(F_\epsilon) = \prod_{m\geq 0} A(q^{-1}, q^k \bar{\epsilon}^{4m+2} x) &= \prod_{m,n\geq 0} \frac{(1 - xq^k \bar{\epsilon}^{4m+2} q^{4n})(1 - xq^k \bar{\epsilon}^{4m+2} q^{4n+4})}{(1 - xq^k \bar{\epsilon}^{4m+2} q^{4n+2})^2} \\ &= \frac{(q^k \bar{\epsilon}^2 x; q^4, \bar{\epsilon}^4)_\infty (q^{k+4} \bar{\epsilon}^2 x; q^4, \bar{\epsilon}^4)_\infty}{(q^{k+2} \bar{\epsilon}^2 x; q^4, \bar{\epsilon}^4)^2_\infty}. \end{aligned} \tag{9.11}$$

and

$$a \pm \hat{d} = \prod_{m\geq 1} \frac{(1 \pm q^{-1+k/2} \sqrt{x}\, \bar{\epsilon}^{2m-1})}{(1 \pm q^{1+k/2} \sqrt{x}\, \bar{\epsilon}^{2m-1})}, \quad b \pm \hat{c} = \prod_{m\geq 1} \frac{(1 \pm q^{-1+k/2} \sqrt{x}\, \bar{\epsilon}^{2m})}{(1 \pm q^{1+k/2} \sqrt{x}\, \bar{\epsilon}^{2m})},$$

with $\bar{\epsilon}^2 = \epsilon^2 q^{-k}$. Finally, we give the result of projecting the universal elliptic R-matrix of $\widehat{\mathfrak{sl}(2)}$ on the evaluation representation ($k = 0$),

$$R_\epsilon(z_1, z_2) = \big((F_\epsilon^{\mathrm{t}})^{-1} R F_\epsilon\big)(z_1, z_2) = A_\epsilon(q, x^{-1}) \begin{pmatrix} \alpha & & & \delta \\ & \beta & \gamma & \\ & \gamma & \beta & \\ d & & & \alpha \end{pmatrix},$$

where

$$\alpha + \delta = q \frac{\theta_3(u - \rho, \tau)}{\theta_3(u + \rho, \tau)}, \quad \alpha - \delta = q \frac{\theta_2(u - \rho, \tau)}{\theta_2(u + \rho, \tau)},$$
$$\beta + \gamma = \frac{\theta_1(u - \rho, \tau)}{\theta_1(u + \rho, \tau)}, \quad \beta - \gamma = \frac{\theta(u - \rho, \tau)}{\theta(u + \rho, \tau)},$$

with

$$x = z_1/z_2 = \mathrm{e}^{4\pi\mathrm{i}u}, \quad q = \mathrm{e}^{2\pi\mathrm{i}\rho}, \quad \epsilon = \mathrm{e}^{\pi\mathrm{i}\tau}, \quad A_\epsilon(q, x^{-1}) = A(q, x^{-1})A(F_\epsilon)/A(F_\epsilon^{\mathrm{t}}).$$

In terms of the Jacobian elliptic functions one has

$$\alpha + \delta : \alpha - \delta : \beta + \gamma : \beta - \gamma = \frac{\mathrm{dn}(2K(u-\rho), k)}{\mathrm{dn}(2K(u+\rho), k)} : 1 : \frac{1}{q}\frac{\mathrm{cn}(2K(u-\rho), k)}{\mathrm{cn}(2K(u+\rho), k)} : \frac{1}{q}\frac{\mathrm{sn}(2K(u-\rho), k)}{\mathrm{sn}(2K(u+\rho), k)},$$

where $K, k$ are the real quarter-period and modulus, respectively, for the nome $\epsilon$:

$$K = \frac{\pi}{2}\prod_{n\geq 1}\left(\frac{1+\epsilon^{2n-1}}{1-\epsilon^{2n-1}} \cdot \frac{1-\epsilon^{2n}}{1+\epsilon^{2n}}\right)^2, \quad k = 4\sqrt{\epsilon}\prod_{n\geq 1}\left(\frac{1+\epsilon^{2n}}{1+\epsilon^{2n-1}}\right)^4.$$


## Acknowledgements.

We thank Olivier Babelon, Benjamin Enriquez, Moshe Flato, Tetsuji Miwa and Nikolai Reshetikhin for advice. We thank Moshe Flato for an incisive and constructive criticism of the original manuscript. A.G. thanks the Fundación Del Amo for financial support and the Department of Physics of UCLA for hospitality.


## Appendix.

*Solving the recursion relations.*

We shall solve the recursion relation (2.4) in the fundamental evaluation representation of $\widehat{\mathfrak{sl}(2)}$. Here we set

$$e_1 = \kappa\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_{-1} = \kappa\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \varphi = \frac{1}{2}H \otimes H, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.1}$$

The commutation relations hold with $\kappa^2 = q - q^{-1}$. The factor $T$ in $R = q^\varphi T$ has the form

$$T = \begin{pmatrix} a & & & \\ & b & cx & \\ & c & b & \\ & & & a \end{pmatrix}, \quad x = z_1/z_2,$$

and (2.4) is equivalent to

$$[T, 1 \otimes e_{-\gamma}] = \big(e_{-\gamma} \otimes q^{\varphi(\gamma,.)}\big)T - T\big(e_{-\gamma} \otimes q^{-\varphi(.,\gamma)}\big), \quad \gamma = 1, 0,$$

with $\varphi(1,.) = \varphi(.,1) = H, \varphi(0,.) = \varphi(.,0) = -H$. Taking $\gamma = 1$ we get two relations,

$$q(a - b) = c = (aq - b/q)/x, \tag{A.2}$$

and taking $\gamma = 0$ the same two relations. Hence

$$R(q, x^{-1}) = \frac{A(q, x^{-1})}{1 - q^{-2}x^{-1}} q^\varphi\big((1 - q^{-2}x^{-1})H_+ + (1 - x^{-1})H_- + e_{-1} \otimes e_1 + e_{-0} \otimes e_0\big).$$

The matrices in (9.4) and (9.5) are found in the same way. In the special case of $\widehat{\mathfrak{sl}(2)}$ the Cartan factors in (7.5) are, for $m \geq 1$,

$$Q(2m,1) = q^{(u-m)c}, \quad Q(2m,0) = q^{(1-u-m)c}, \quad Q(2m-1,1) = Q(2m-1,0) = q^{(1-m)c}.$$

In the structure, $R$ is determined uniquely by the recursion relations and the initial conditions, but in the evaluation representation the normalizing factor $A(q,x^{-1})$ remains undetermined. Fortunately Levendorskii, Soibelman and Stukopin [LS], starting from an equivalent expression for the standard, universal R-matrix for $\widehat{\mathfrak{sl}(2)}$ obtain the following result,

$$A(q,x) = \exp\Big(\sum_{k\geq 1} \frac{1}{k}\frac{q^k - q^{-k}}{q^k + q^{-k}} x^k\Big). \tag{A.3}$$

The sum converges for $|q| \neq 1, |x| < 1$ and the formula can be manipulated to yield

$$A(q,x) = \begin{cases} \dfrac{(xq^2;q^4)_\infty^2}{(x;q^4)_\infty (xq^4;q^4)_\infty}, & |q| < 1, \\ \dfrac{(x;q^{-4})_\infty (xq^{-4};q^{-4})_\infty}{(xq^{-2};q^{-4})_\infty^2}, & |q| > 1. \end{cases} \tag{A.4}$$

Hence

$$A(q,x)A(q^{-1},x) = 1, \quad |q| \neq 1. \tag{A.5}$$

This is also clear from (A.3).

The inverse of $R$ can also be represented as a series, similar to (2.1),

$$R^{-1} = q^{-\varphi}\hat{T}, \quad \hat{T} = 1 + \sum_\alpha \hat{e}_{-\alpha} \otimes \hat{e}_\alpha + \dots,$$

with

$$\hat{e}_\alpha := q^{-\varphi(\alpha,.)} e_\alpha, \quad \hat{e}_{-\alpha} = -e_{-\alpha} q^{\varphi(.,\alpha)}.$$

The commutation relations for the $\hat{e}$'s agree with those of the $e$'s, and the recursion relations for $\hat{T}$ agrees with that of $T$, all up to the sign of $\varphi$. (We get a recursion relation for $\hat{T}$ from the fact that $R^{-1}$ also satisfies the Yang-Baxter relation.) Consequently, in the structure,

$$R(\varphi, e)^{-1} = R(-\varphi, \hat{e}),$$

and in any evaluation representation,

$$R(q,x)^{-1} = R(q^{-1}, x).$$

These results are quite general and imply, in particular, Eq.(A.5).

*Reduced formulas.*

We list here the formulas that are obtained from the Yang-Baxter relation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

and the quasi triangular conditions

$$(\mathrm{id} \otimes \Delta)R = R_{12}R_{13}, \quad (\Delta \otimes \mathrm{id})R = R_{23}R_{13}$$

when the $c, d$ factors are removed as in

$$R = q^{uc \otimes d + (1-u) d \otimes c} \tilde{R},$$

namely

$$\tilde{R}_{12}(q^{-uc_2 d_3} \tilde{R}_{13} q^{uc_2 d_3}) \tilde{R}_{23} = \tilde{R}_{23}(q^{-(1-u) d_1 c_2} \tilde{R}_{13} q^{(1-u) d_1 c_2}) \tilde{R}_{12} \tag{A.6}$$

and

$$(\mathrm{id} \otimes \Delta)\tilde{R} = (q^{-(1-u) d_1 c_3} \tilde{R}_{12} q^{(1-u) d_1 c_3}) \tilde{R}_{13}, \quad (\Delta \otimes \mathrm{id})\tilde{R} = (q^{-uc_1 d_3} \tilde{R}_{23} q^{uc_1 d_3}) \tilde{R}_{13}. \tag{A.7}$$

These last two relations give us what we need to reduce (8.8), namely

$$L^-(z_1)_{13} \Phi(z_2) = \big(\tilde{R}_{12}(\frac{z_2}{z_1} q^{k-uk})\big)^{-1} \big(L^{-i}(z_1) \otimes 1\big) \Phi(z_2) L_i^-, \tag{A.8}$$

$$L^{+i}(z_2) \Phi(z_1) L_i^+ = L^+(z_2) \tilde{R}_{12}(\frac{z_2}{z_1}) \Phi(z_1). \tag{A.9}$$

For the other intertwiner, there is an analogue of (A.9),

$$L^{+i}(z_2) \Psi(z_1) L_i^+ = \tilde{R}_{12}(\frac{z_2}{z_1} q^{-uk}) L^+(z_2) \Psi(z_1), \tag{A.10}$$

but we could not find an analogue of (A.8). To obtain (8.14) we used the method that was explained for the derivation of (8.21).

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